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Effective theory of magnetization plateaus in a three-leg ladder with periodic boundary conditions

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Abstract. We show that the low-energy physics of the three-leg-ladder model in the presence of a critical magnetic field can be described by a broken SU(3) spin chain for a periodic rung model and a broken SU(2) spin chain for an open rung model. Using the Lieb–Schultz–Mattis theorem we characterize the possible magnetization plateaus and study the critical behaviour in the region of transition between the plateaus $\langle S^z \rangle = 1/2$ and $\langle S^z \rangle = 3/2$ by means of renormalization group calculations performed on the bosonized effective continuum field theory. We show that in certain regions of the parameter space of the effective theory the system remains gapless, and we compute the spin–spin correlation functions for these regions. We also discuss the possibility of a plateau at $\langle S^z \rangle = 1$, and show that although there exists in the continuum theory a term that might cause the appearance of a plateau there, such a term is unlikely to be relevant. This conjecture is proved by density-matrix renormalization group (DMRG) techniques. The modifications of the three-leg-ladder Hamiltonian that show plateaus at $\langle S^z \rangle = 1, 5/6, 7/6$ are discussed. We show the $\langle S^z \rangle = 1$ plateau in the XXZ-type modification by means of the DMRG technique. We find non-magnetic gapless excitation on this plateau in the periodic rung case.

1. Introduction

One-dimensional and quasi-one-dimensional quantum spin systems have attracted much attention in recent years due to the large number of experimental realizations of such systems and the variety of theoretical techniques, both analytical and numerical, available for studying the relevant models. Due to the presence of large quantum fluctuations in low dimensions, these systems present unusual properties such as a gap between a singlet ground state and excited non-singlet states. Examples include spin-ladder systems in which a small number of one-dimensional spin-1/2 chains interact among themselves [1]. In this case, in a way very similar to that in the Haldane spin-S problem [2], it has been found that if the number of chains is even, the system effectively behaves as an integer-spin chain with a gap in the low-energy spectrum, while it remains massless for an odd number of chains. Some two-chain ladders which exhibit a gap are $SrCu_2O_3$ [3] and $Cu_2(C_5H_{12}N_2)_2Cl_4$ [4], and an example of a gapless three-chain ladder is $Sr_2Cu_3O_5$ [3]. Thus far we have implicitly assumed that the boundary conditions in the transverse direction are open boundary conditions (OBC). These boundary conditions

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correspond to having all the chains lying in the same plane. This is the situation encountered in experimental systems such as $Sr_2Cu_3O_5$. In contrast with OBC, periodic boundary conditions (PBC) are frustrating for $(2n+1)$ coupled spin chains. As a consequence all the spin excitations are gapped [5, 6] in the case of periodic boundary conditions. They are also gapped for $2n$ coupled spin chains with PBC but the mechanism is related to singlet formation as in the OBC case and not frustration. The PBC could be achieved in an experimental system by having the coupled chains forming a cylinder instead of lying in a plane.

A richer behaviour emerges when these gapped or ungapped systems are placed in a magnetic field. Then it is possible for an integer-spin chain to be gapless and a half-odd-integer-spin chain to show a gap above the ground state for certain values of the field [7–10]. This has been demonstrated by several methods such as bosonization [5, 11], perturbation theory [12], and the density-matrix renormalization group method (DMRG) [13–15]. In particular, it has been shown that spin-1/2 chains and ladders with a gap undergo continuous phase transition from a commensurate zero-uniform-magnetization phase to an incommensurate phase with non-zero magnetization [10], and the magnetization of the system can exhibit plateaus at certain non-zero values of the magnetic field [8, 16]. Furthermore, a striking property of the quantum spin chains in a uniform magnetic field pointing along the direction of the axial symmetry (the z-direction) is the topological quantization of the magnetization under a changing of the magnetic field [7]. It was shown, starting from a generalized Lieb–Schultz–Mattis (LSM) theorem [17], that translationally invariant spin chains in an applied field can be gapful without breaking translational symmetry only when the magnetization per spin, m, obeys $S - m =$ integer, where S is the maximum possible spin in each unit cell of the Hamiltonian [7–9]. Such gapped phases correspond to plateaus at these quantized values of m. In reference [18] the behaviour of the magnetization versus magnetic field has been investigated in detail using DMRG techniques for three coupled spin-1/2 chains with both periodic and open boundary conditions. Plateaus have been obtained at $m = 1/2$ and $m = 3/2$ in agreement with reference [7]. Furthermore, for the case of PBC a small plateau at $m = 0$ was also obtained [9, 15, 18]. Finally, there seems to exist some weak evidence for a plateau at $m = 1$ for PBC [18]. Strong-coupling low-energy Hamiltonians for these two systems were also derived in reference [18].

In this paper, we investigate a three-leg ladder with PBC (spin tube) and with OBC in the presence of a uniform magnetic field by using bosonization and renormalization group techniques. We are concerned with the transition region between the magnetization plateaus at $m = 1/2$ and $m = 3/2$. Our analysis is based on the low-energy effective Hamiltonian (LEH) derived for strong coupling between the rungs [18]. We identify the LEH as an anisotropic SU(3) spin chain with symmetry-breaking terms in a longitudinal magnetic field, and analyse its low-energy physics via bosonization and RG techniques. This approach allows us to predict the behaviour of the spin–spin correlation functions in this transition region and the NMR relaxation rate. This also allows an investigation of the possibility of a non-trivial plateau at $m = 1$. In the XXZ three-leg-ladder model, we show a plateau at $m = 1$. In the PBC case, there is a non-magnetic gapless excitation. In the OBC case, the excitations are all gapped.

The paper is organized as follows. In section 2, we recall the derivation [18] of the LEH and reduce it to an anisotropic SU(3) spin chain, while that for the OBC ladder reduces to an anisotropic SU(2) spin chain. The difference between PBC and OBC models becomes obvious in the language of effective Hamiltonians. We review briefly the analysis of the magnetization process of the isotropic SU(3) spin chain, and discuss the possibility of a cusp in the magnetization process of the three-leg ladder with OBC. The bosonized Hamiltonian is derived in section 3. In section 4, we analyse the low-energy effective Hamiltonian in a weak-coupling limit by calculating the one-loop renormalization group (RG) in the marginally perturbed SU(3) Wess–Zumino–Witten model [19] and discuss the renormalization group flow. For weak coupling, the flow is to an invariant surface, leading to gapless excitations above the ground state with no breaking of the discrete symmetry. On the basis of the weak-coupling renormalization group analysis and the usual continuity between weak coupling and strong coupling in one-dimensional systems, we claim that the spin tube is described by a two-component Luttinger liquid at low energy and long wavelength. In section 5, we discuss the effect of a variation of the magnetic field in that problem, and show that it does not affect the two-component Luttinger liquid behaviour. We discuss the analogy of this two-component Luttinger liquid with the S2 phase of the bilinear–biquadratic spin-1 chain [20]. Then, having established the equivalence with the two-component Luttinger liquid, we calculate the spin-correlation functions in the critical region and the temperature dependence of the NMR longitudinal relaxation rate T_1^{-1}. We also present a theoretical description of the plateau at $m = 1$ in the framework of bosonization. Comparing this description with the numerical results of reference [18] we conclude that the presence of a plateau at $m = 1$ is unlikely in the spin tube. We verify our results on the absence of plateaus at $m = 1$ using DMRG theory, and indicate the XXZ modifications of the three-leg-ladder Hamiltonian that could lead to a plateau. Section 6 contains the concluding remarks. Technical details can be found in the appendices.

2. The low-energy effective Hamiltonian of the spin tube

The Hamiltonian of the three-chain ladder with periodic boundary conditions (PBC) in the presence of an external magnetic field is

$$H = J\sum_{i=1}^{N}\sum_{p=1}^{3}\vec{S}_{i,p}\cdot\vec{S}_{i+1,p} + J_\perp\sum_{i=1}^{N}\sum_{p=1}^{3}\vec{S}_{i,p}\cdot\vec{S}_{i,p+1} - \vec{h}\cdot\sum_{i=1}^{N}\sum_{p=1}^{3}\vec{S}_{i,p} \qquad (2.1)$$

where p (i) is a chain (site) index, J is the coupling along the chain, $J_\perp$ is the transverse coupling, and the site $(i, 4)$ is identified with the site $(i, 1)$. The three-chain ladder with periodic boundary conditions can be viewed as forming a tube with an equilateral triangular cross section (see figure 1). We will refer to this system as a *spin tube*.

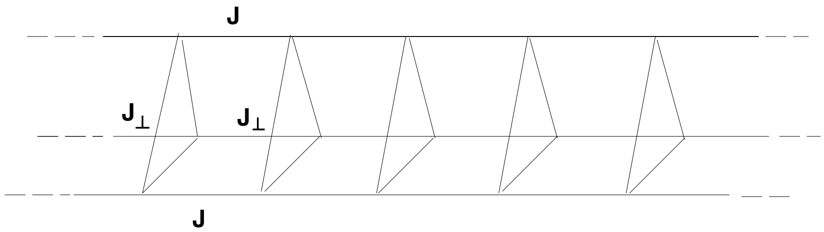

Figure 1. The cylindrical three-leg ladder (spin tube). The choice of the topology affects the strong-coupling limit.

In the rest of the paper we shall consider the model for $J_\perp \gg J$, and the aim of this section is to recall briefly the derivation of the low-energy Hamiltonian [18] in this limit. To begin with, for $J = 0$, the system consists of independent rungs. The eight states of a given rung fall into a spin-3/2 quadruplet and two spin-1/2 doublets. In the absence of a magnetic field, the spin-3/2 states on a given triangle are all degenerate with energy $(3/4)J_\perp$. These states are

$$
\begin{aligned}
|3/2; 3/2\rangle &= |\uparrow\uparrow\uparrow\rangle \\
|3/2; 1/2\rangle &= \frac{1}{\sqrt{3}}[|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle] \\
|3/2; -1/2\rangle &= \frac{1}{\sqrt{3}}[|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle] \\
|3/2; -3/2\rangle &= |\downarrow\downarrow\downarrow\rangle.
\end{aligned}
\tag{2.2}
$$

Also, in the absence of a magnetic field and on a given rung, the two spin-1/2 doublets, corresponding to the left and right chiralities $(-/+)$, are degenerate with energy $-(3/4)J_\perp$. These states are

$$
\begin{aligned}
|\uparrow+\rangle &= \frac{1}{\sqrt{3}}[|\downarrow\uparrow\uparrow\rangle + j|\uparrow\downarrow\uparrow\rangle + j^2|\uparrow\uparrow\downarrow\rangle] \\
|\downarrow+\rangle &= \frac{1}{\sqrt{3}}[|\uparrow\downarrow\downarrow\rangle + j|\downarrow\uparrow\downarrow\rangle + j^2|\downarrow\downarrow\uparrow\rangle] \\
|\uparrow-\rangle &= \frac{1}{\sqrt{3}}[|\downarrow\uparrow\uparrow\rangle + j^2|\uparrow\downarrow\uparrow\rangle + j|\uparrow\uparrow\downarrow\rangle] \\
|\downarrow-\rangle &= \frac{1}{\sqrt{3}}[|\uparrow\downarrow\downarrow\rangle + j^2|\downarrow\uparrow\downarrow\rangle + j|\downarrow\downarrow\uparrow\rangle]
\end{aligned}
\tag{2.3}
$$

where $j = \exp(2\pi \mathrm{i}/3)$.

When an external magnetic field is switched on, the degeneracy in the different multiplets is lifted. The energy levels of the state $|\uparrow\uparrow\uparrow\rangle$ (in the spin-3/2 multiplet) and the spin-1/2 states $|\uparrow+\rangle$, $|\uparrow-\rangle$ cross at $h_c = \frac{3}{2}J_\perp$ (see figure 2). As a result, for $h < h_c$, one has a ground-state magnetization $\langle S^z\rangle = 1/2$, and for $h > h_c$, $\langle S^z\rangle = 3/2$, i.e. h_c is a transition point between two magnetization plateaus. If a small coupling J is turned on, this transition is expected to broaden between $h_{1/2,+}$ and $h_{3/2,-}$, where $h_{3/2,-} - h_{1/2,+}$ is of the order of J. We expect that in this interval $\langle S^z\rangle$ will increase continuously with h. In this limit the properties of the system can be studied by perturbing with H_1 around the decoupled rung Hamiltonian H_0:

$$H = H_0 + H_1 \tag{2.4}$$

$$H_0 = J_\perp \sum_{i=1}^{N}\sum_{p=1}^{3} \vec{S}_{i,p}\cdot\vec{S}_{i,p+1} - h_c \sum_{i=1}^{N}\sum_{p=1}^{3} S^z_{i,p} \tag{2.5}$$

$$H_1 = J \sum_{i=1}^{N}\sum_{p=1}^{3} \vec{S}_{i,p}\cdot\vec{S}_{i+1,p} - (h - h_c) \sum_{i=1}^{N}\sum_{p=1}^{3} S^z_{i,p}. \tag{2.6}$$

At $h = h_c$ the ground state of H_0 is 3^N-fold degenerate, the states $|\uparrow-\rangle_i$, $|\uparrow+\rangle_i$, $|\uparrow\uparrow\uparrow\rangle_i$ (to be denoted respectively as $|\tilde{1}\rangle_i$, $|\tilde{2}\rangle_i$, $|\tilde{3}\rangle_i$) spanning the low-energy subspace. H_1 lifts the degeneracy in the subspace, leading to an effective Hamiltonian that can be derived by standard perturbation theory. Since in the truncated subspace there are three states per triangle, it is natural to express the spin operators in the basis given by Gell-Mann matrices λ^α, $\alpha = 1, \ldots, 8$. (The conventions that we use for the matrices can be found for instance in references [21] and [22].) By considering the action of the spin operators $S^+_{1,2,3}$ and $S^z_{1,2,3}$ on each state of the truncated Hilbert space, the spin operators can be expressed in terms of the matrices as

$$S^+_{i,p} = \frac{1}{2\sqrt{3}}[j^{p-1}(\lambda^6_i + \mathrm{i}\lambda^7_i) + j^{2(p-1)}(\lambda^4_i - \mathrm{i}\lambda^5_i)] \tag{2.7}$$

$$S^z_{i,p} = \frac{1}{3}\left[\frac{5}{6}I - \frac{\lambda^8_i}{\sqrt{3}} - j^{2(p-1)}(\lambda^1_i + \mathrm{i}\lambda^2_i) - j^{(p-1)}(\lambda^1_i - \mathrm{i}\lambda^2_i)\right]. \tag{2.8}$$

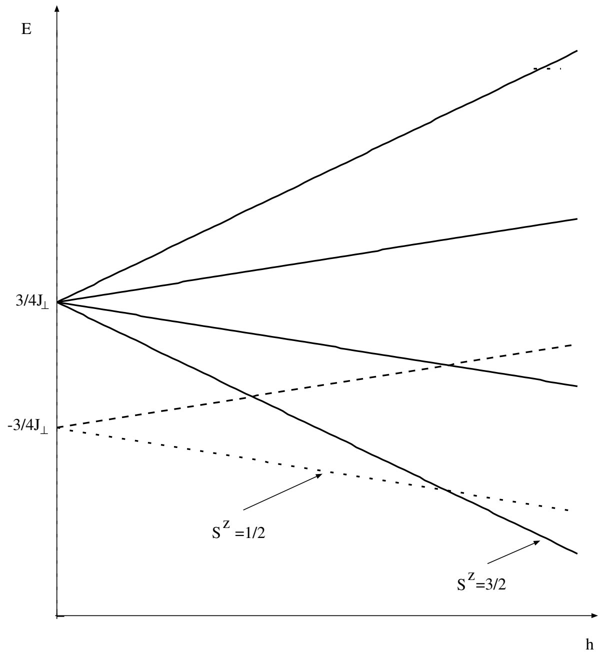

Figure 2. The energy levels of a single triangle as a function of the magnetic field. Solid lines correspond to states with $S = 3/2$, dashed lines to states with $S = 1/2$. One observes the level crossing between the state with $S^z = 3/2$ and the states with $S = 1/2$, $S^z = 1/2$ as the magnetic field is increased.

where I is the identity matrix. The total rung spin is given by

$$S_i^z = \left(\frac{5}{6}I - \frac{\lambda_i^8}{\sqrt{3}}\right). \tag{2.9}$$

The effective Hamiltonian to first order then becomes

$$H_{eff} = \tilde{H}_0 + \tilde{H}_I \tag{2.10}$$

$$\tilde{H}_0 = \frac{J}{4}\sum_{i=1}^{N}\sum_{\alpha=1}^{8}\lambda_i^\alpha\lambda_{i+1}^\alpha \tag{2.11}$$

$$\tilde{H}_I = q\sum_{i=1}^{N}[\lambda_i^1\lambda_{i+1}^1 + \lambda_i^2\lambda_{i+1}^2] + u\sum_{i=1}^{N}\lambda_i^3\lambda_{i+1}^3 + u'\sum_{i=1}^{N}\lambda_i^8\lambda_{i+1}^8 + \frac{h_{eff}}{\sqrt{3}}\sum_{i=1}^{N}\lambda_i^8. \tag{2.12}$$

In our case, $q = -J/12$, $u = -J/4$, $u' = -5J/36$, and $h_{eff} = h - h_c - 5J/9$; hereafter we choose our units such that $J = 1$. The Hamiltonian (2.10) is written as an isotropic SU(3) spin chain $\tilde{H}_0$ and terms $\tilde{H}_I$ that break the symmetry. This form will be convenient later on when we study such questions as what regions of parameter space are gapless and the behaviour of correlations functions there.

Another form of the Hamiltonian is convenient when one wishes to study the plateau structure. We introduce [18] a different basis of SU(3) operators $T_1^\pm$, $T_2^\pm$, $T_3^\pm$, and T^z defined by

$$T_1^\pm = (\lambda_1 \pm \mathrm{i}\lambda_2)/2 \tag{2.13}$$

$$T_2^\pm = (\lambda_4 \pm \mathrm{i}\lambda_5)/2 \tag{2.14}$$

$$T_3^{\pm} = (\lambda_6 \pm \mathrm{i}\lambda_7)/2 \tag{2.15}$$

$$T^z = -2\frac{\lambda_8}{\sqrt{3}}. \tag{2.16}$$

Then, to first order, and up to a constant, the effective Hamiltonian reads

$$\begin{aligned} H_{eff} = \frac{J}{2}\sum_i [T^+_{i,2}T^-_{i+1,2} + T^-_{i,2}T^+_{i+1,2} + T^+_{i,3}T^-_{i+1,3} + T^-_{i,3}T^+_{i+1,3}] \\ + \frac{J}{3}\sum_i [T^+_{i,1}T^-_{i+1,1} + T^-_{i,1}T^+_{i+1,1}] + \frac{J}{12}\sum_i T^z_i T^z_{i+1} \\ - \left(\frac{1}{2}h - \frac{3}{4}J_\perp - \frac{5}{18}J\right)\sum_i T^z_i. \end{aligned} \tag{2.17}$$

This is the Hamiltonian derived in reference [18]. In this form the underlying structure of an anisotropic SU(3) spin chain in a 'λ^8 magnetic field' is unexploited. The correspondence between our notation and that of reference [18] can be found in table 1.

Table 1. The correspondence between the notation of the present paper and that of Tandon *et al*.

	States			
Present paper	$\vert\tilde{1}\rangle$	$\vert\tilde{2}\rangle$	$\vert\tilde{3}\rangle$	
Tandon *et al*	$j\vert 7'\rangle$	$j^2\vert 5'\rangle$	$\vert 1\rangle$	
	Operators			
Present paper	T_1^+	T_2^+	T_3^+	$T^z + 1/3$
Tandon *et al*	$j^2\tau^-$	jL^-	j^2R^-	σ^z

The form of the Hamiltonian (2.10) may help in relating our model to integrable versions of the SU(3) spin chains. Isotropic spin chains are known to be integrable by Bethe *ansatz* techniques [23, 24]. The magnetization process of SU(3) spin chains with a magnetic field coupled to λ^3 or λ_8 has been analysed in the context of the bilinear–biquadratic spin-1 chain at the integrable Uimin–Lai–Sutherland point [25–27] by solving numerically the Bethe *ansatz* equations. There exist also integrable anisotropic SU(3) spin chains [28], but the chain described by the Hamiltonian (2.10) is not one of them. On the basis of the results already known and some simple arguments, we can however obtain a qualitative picture of the magnetization process. First, by applying a straightforward generalization of the Lieb–Schultz–Mattis theorem to the SU(3) chains to the effective Hamiltonian equation (2.6), one can show that the magnetization plateaus predicted using the effective Hamiltonian (2.6) are identical to those predicted using the original Hamiltonian of the spin tube. In the study of the magnetization process of an *isotropic* SU(3) chain in reference [25, 26] a cusp in the magnetization process was found for a critical magnetic field. Such a cusp is not related to plateau formation. One may thus wonder whether a cusp could also be obtained in the magnetization process of the chain described by Hamiltonian (2.17). However, it is important to stress that in references [25,26], the magnetic field was coupled to λ_3, and the cusp resulted from the emptying of the band of excitations carrying $\lambda_3 = -1$ for large enough field. In the case that we investigate, the field is coupled to λ_8. Studies of the magnetization process [27] with a magnetic field coupled to λ_8 showed no such cusp in the curve of $\langle\lambda_8\rangle$ versus h. The reason for this is that the elementary excitations have respective chemical potentials $-h$, $-h$, $2h$, so no band can become empty before a fully polarized state is reached. By analogy with the latter case, we should not expect any cusp in the magnetization process of the system described by

Hamiltonian (2.17). As a result, we expect the magnetization plateaus obtained in the study of the magnetization process of the Hamiltonian (2.17) to be those of the SU(3) spin chain and that no cusp should appear in the magnetization process.

To make more detailed statements on the plateau formation and the spin–spin correlation functions of the ladder we will have to resort to a combination of approximate methods such as bosonization and renormalization group techniques. This is the object of section 3.

We conclude this section by contrasting the open and periodic boundary conditions. The same strong-coupling analysis can be done for the OBC case. In contrast with the PBC case, we have only a twofold degeneracy instead of a threefold one at $J = 0$ under a strong field $h = 7J_\perp/8$. These two low-energy states are

$$\begin{aligned} |3/2; 3/2\rangle &= |\uparrow\uparrow\uparrow\rangle \\ |1/2; 1/2\rangle &= \sqrt{\frac{2}{3}}\left(\frac{1}{2}|\uparrow\uparrow\downarrow\rangle + \frac{1}{2}|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle\right). \end{aligned} \tag{2.18}$$

The effective Hamiltonian to first-order perturbation in $J/J_\perp$ becomes the well known spin-1/2 XXZ model:

$$H_{eff} = \frac{J}{4}\sum_i(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z) - \left(\frac{h}{2} - \frac{7J_\perp}{16}\right)\sum_i \sigma_i^z \tag{2.19}$$

where $\Delta = \frac{5}{18}$. It is well known that this Hamiltonian has no gap for $\Delta \leqslant 1$ except at the saturated magnetization $\sigma^z = \pm 1$, which corresponds to $m = 1/2, 3/2$ in the original ladder model. This agrees with a weak-coupling analysis ($J \gg J_\perp$) based on bosonization [5, 9]. In the strong-coupling analysis, one can clearly see the difference between PBC and OBC in their effective Hamiltonian.

3. Bosonization and weak-coupling analysis

We proceed now to study the long-distance properties of the effective Hamiltonian H_{eff} defined in equation (2.10). It is a sum of an isotropic SU(3) spin-chain Hamiltonian plus SU(3)-symmetry-breaking terms:

$$H_{eff} = \tilde{H}_0 + \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3 + \tilde{H}_h \tag{3.1}$$

with

$$\tilde{H}_0 = \frac{J}{4}\sum_{i=1}^{N}\sum_{\alpha=1,\ldots,8}\lambda_i^\alpha\lambda_{i+1}^\alpha \tag{3.2}$$

$$\tilde{H}_1 = q\sum_{i=1}^{N}[\lambda_i^1\lambda_{i+1}^1 + \lambda_i^2\lambda_{i+1}^2] \tag{3.3}$$

$$\tilde{H}_2 = u\sum_{i=1}^{N}\lambda_i^3\lambda_{i+1}^3 \tag{3.4}$$

$$\tilde{H}_3 = u'\sum_{i=1}^{N}\lambda_i^8\lambda_{i+1}^8 \tag{3.5}$$

$$\tilde{H}_h = h_{eff}\sum_{i=1}^{N}\frac{\lambda_i^8}{\sqrt{3}}. \tag{3.6}$$

In our case, $q = -(1/12)J$, $u = -(1/4)J$, $u' = -(5/36)J$, and $h_{eff} = (h - h_c)/2 - (5/18)J$; hereafter we choose our units such that $J = 1$.

3.1. Non-Abelian bosonization of an SU(3) spin chain

The SU(3)-invariant Hamiltonian $\tilde{H}_0$ can be solved exactly by the Bethe *ansatz* [23, 24]. The solution shows that the SU(3) spin chain has two branches of excitations, with dispersion

$$\epsilon_j(k) = \frac{J}{4}\frac{2\pi}{\sin(\pi j/3)}[\cos(\pi j/3 - |k|) - \cos(\pi j/3)] \qquad j = 1, 2.$$

These excitations are gapless, and for $|k| \to 0$, one has

$$\epsilon_1(k) = \epsilon_2(k) \simeq \frac{2\pi}{3}\frac{J}{4}|k|$$

i.e. the dispersion relation assumes at long wavelength a massless relativistic form. Accordingly, the low-energy, long-wavelength excitations of the SU(3) spin chain can be bosonized. More precisely, these excitations are described [29, 30] by the SU(3) level-1 ($SU(3)_1$) Wess–Zumino–Novikov–Witten (WZNW) model [31], perturbed by a marginally irrelevant SU(3)-invariant operator. A review of WZNW models can be found in reference [32]. In Hamiltonian form, the $SU(3)_1$ model can be written as

$$H_{\mathrm{WZNW}} = \frac{2\pi}{3}\sum_{a=1}^{3^2-1} {:}J_R^a(x)J_R^a(x){:} + {:}J_L^a(x)J_L^a(x){:} \tag{3.7}$$

where the right and left currents satisfy the following commutation relations (Kac–Moody algebra at level 1):

$$[J^\alpha_{R(L)}, J^\beta_{R(L)}] = \mathrm{i}f^{\alpha\beta\gamma}\delta(x-y)J^\gamma_{R(L)}(y) + \frac{\mathrm{i}\delta_{\alpha\beta}}{2\pi}\delta'(x-y). \tag{3.8}$$

In equation (3.8), the $f^{\alpha\beta\gamma}$ are the structure constants of SU(3). The central charge is $C = 1 \times (3^2-1)/(3+1) = 2$, indicating that the $SU(3)_1$ WZNW model can be described in terms of two free-bosonic fields. As mentioned above, the SU(3) spin chain is described asymptotically by the $SU(3)_1$ model perturbed by a marginally irrelevant SU(3)-invariant term:

$$H \to H_{\mathrm{WZW}} + g_0\int\frac{\mathrm{d}x}{2\pi}\,\Phi_0(x) \tag{3.9}$$

where the marginal operator

$$\Phi_0(x) = \sum_{\alpha=1}^{3} J_R^\alpha(x)J_L^\alpha(x)$$

couples the right and left currents.

The finite-size correction to the ground-state energy of the SU(3) chain can be obtained from the Bethe *ansatz* solution. These corrections are logarithmic and are in agreement with those obtained from the continuum Hamiltonian (3.9). This situation is very similar to the more familiar case of the SU(2) spin chain, which is described at low energy and long wavelength by the marginally perturbed $SU(2)_1$ WZNW model [33]. In general, the magnitude of g_0 cannot be obtained from the lattice Hamiltonian in the case of an SU(N) spin chain (see the discussion of the case of $N = 2$ in reference [33]). This is even more problematic when one adds perturbations to the SU(3)-invariant spin chain. Another difficulty is that these perturbations are not small in our case and strictly speaking cannot be treated in perturbation theory. However, in one dimension weak- and strong-coupling behaviour are often continuously connected [34–37], so a weak-coupling analysis can provide very valuable information on the qualitative physics for strong coupling. Therefore, if we can find a weak-coupling model that is described by the marginally perturbed $SU(3)_1$ WZNW model and if we add to it small perturbations of the form (3.3)–(3.5), we will be able to make a reasonable guess at the low-energy, long-wavelength continuum theory associated with the Hamiltonian H_{eff}. By analogy with the Heisenberg model, we expect the difference between the weak- and the strong-coupling regime to reduce to a renormalization of some parameters of the effective low-energy theory. For non-integrable models, these parameters can be obtained numerically by calculating *thermodynamic* quantities via exact-diagonalization methods [38–40].

In our case, it is not difficult to see that the spin sector of the SU(3) Hubbard model [30,41] is a good candidate for a weak-coupling model. This model is defined by the Hamiltonian

$$H = -t \sum_{i,n=1,2,3} (c^{\dagger}_{i+1,n} c_{i,n} + \text{h.c.}) + U \sum_{i,n\neq m} n_{i,n} n_{i,m} \tag{3.10}$$

where $c_{i,n}$ annihilates a fermion of flavour $n \in [1, 2, 3]$ at site i, and $n_{i,n} = c^{\dagger}_{i,n} c_{i,n}$. The basic idea is that, starting from the lattice Hamiltonian of the SU(3) Hubbard model, it is possible to take the continuum limit and then separate the spin excitations from the charge excitations by means of weak-coupling bosonization. In the strong-coupling limit, $U \to \infty$, a constraint of one fermion per site is imposed:

$$\sum_n c^{\dagger}_{i,n} c_{i,n} = 1. \tag{3.11}$$

With one fermion per site, the charge degrees of freedom are frozen out and one is left just with SU(3) spin degrees of freedom.

Second-order perturbation theory in t then shows that the model can be mapped onto an isotropic SU(3) spin chain with the lattice SU(3) spin operators

$$\Lambda^{\alpha}_{i} = \sum_{n,m} c^{\dagger}_{i,m} \lambda^{\alpha}_{n,m} c_{i,n} \tag{3.12}$$

under the constraint (3.11). Under the hypothesis of continuity, the same $\mathrm{SU}(3)_1$ field theory should describe the weak- and strong-coupling limits in the spin sector. The difference between the weak- and strong-coupling limits corresponds to the disappearance of the charge sector. This reduction of the number of degrees of freedom can be obtained in a consistent way by treating the constraint (3.11) within the effective theory [42, 43].

Thus, our strong-coupling theory is the spin sector of the SU(3) Hubbard model with a filling of one fermion per site and $U \to \infty$. Let us discuss the weak-coupling regime. The constraint (3.11) sets the Fermi momentum at $k_F = \pi/3$ for the three fermion flavours. Since we are interested in low-energy, long-wavelength properties, we linearize the spectrum for each flavour around the two Fermi points and introduce the right- and left-moving fermion modes in the continuum limit:

$$c^{\dagger}_{i,n} \simeq \sqrt{a}(\mathrm{e}^{\mathrm{i}k_F x} \psi^{\dagger}_{L,n}(x) + \mathrm{e}^{-\mathrm{i}k_F x} \psi^{\dagger}_{R,n}(x)) \tag{3.13}$$

where $x = ia$ and a is the lattice spacing.

For $U = 0$, the linearized Hamiltonian is

$$H_{\text{linearized}} = -\mathrm{i}v_F \sum_n \int \mathrm{d}x\ (\psi^{\dagger}_{R,n}\, \partial_x \psi_{R,n} - \psi^{\dagger}_{L,n}\, \partial_x \psi_{L,n}). \tag{3.14}$$

This Hamiltonian is conformally invariant and can be rewritten in terms of the right and left charge currents

$$J_{R\,(L)} = \sum_n \psi^{\dagger}_{R\,(L),n} \psi_{R\,(L),n}$$

and the eight SU(3) spin currents (right and left)

$$J^{a}_{R\,(L)} = \sum_n \psi^{\dagger}_{R\,(L),n} \frac{\lambda^{a}_{n,m}}{2} \psi_{R\,(L),m}.$$

One thus separates the charge and spin sectors:

$$H = H_{\text{charge}} + H_{\text{spin}} \tag{3.15}$$

where the charge sector is

$$H_{\text{charge}} = v_F \int \mathrm{d}x \, {:}J_R(x)J_R(x){:} + {:}J_L(x)J_L(x){:} \tag{3.16}$$

and the spin sector is again described by the $SU(3)_1$ model discussed earlier:

$$H_{\text{spin}} = v_F \sum_a \int \mathrm{d}x \, {:}J_R^a(x)J_R^a(x){:} + {:}J_L^a(x)J_L^a(x){:}. \tag{3.17}$$

The charge currents satisfy $U(1)$ Kac–Moody algebra, whereas the spin currents satisfy the $SU(3)_1$ Kac–Moody algebra as can be checked explicitly. When the interaction is weakly turned on, $U/t \ll 1$, it does not break spin–charge separation but induces a $g_0 \propto U$ term [30].

We have discussed thus far non-Abelian bosonization in order to stay close to the literature on SU(3) spin chains. However, an Abelian bosonization approach to the isotropic SU(3) spin chains starting from the SU(3) Hubbard model is perfectly feasible. Such an approach has been introduced for isotropic SU(N) spin chains in reference [41]. It is outlined in appendix A. In fact, for the rest of this section, we shall employ Abelian bosonization because it renders the calculation of correlation functions extremely easy even when the SU(3) symmetry is explicitly broken.

3.2. The Abelian bosonization approach

Abelian bosonization gives the following Hamiltonian for an SU(3)-invariant spin chain (or the spin sector of the SU(3) Hubbard model):

$$\begin{aligned} H_{SU(3)} = & \int \frac{\mathrm{d}x}{2\pi} \, u\left[(\pi\Pi_a)^2 + (\pi\Pi_b)^2 + (\partial_x\phi_a)^2 + (\partial_x\phi_b)^2\right] \\ & + \frac{2Ua}{(2\pi a)^2} \int \mathrm{d}x \, (\cos\sqrt{8}\phi_a + \cos\sqrt{2}(\phi_a + \sqrt{3}\phi_b) + \cos\sqrt{2}(\phi_a - \sqrt{3}\phi_b)) \\ & - Ua \int \frac{\mathrm{d}x}{\pi^2} \, \left[(\partial_x\phi_a)^2 + (\partial_x\phi_b)^2\right]. \end{aligned} \tag{3.18}$$

A derivation can be found in appendix A. The free term corresponds to equation (3.17).

Under renormalization, $H_{SU(3)}$ flows to a fixed-point Hamiltonian [41]:

$$H^* = \int \frac{\mathrm{d}x}{2\pi} \, u^*\left[(\pi\Pi_a)^2 + (\pi\Pi_b)^2 + (\partial_x\phi_a)^2 + (\partial_x\phi_b)^2\right] \tag{3.19}$$

where u^* is given by the Bethe *ansatz* as $u^* = (2\pi/3)J/4$. One can check using expressions (A.12) that this leads to a scaling dimension of 1 for the uniform component of $\Lambda^\alpha(x) \simeq a^{-1}\Lambda_i^\alpha$, $x = ia$ ($\alpha = 1, \ldots, 8$), and 2/3 for the $2\pi/3$ component (see equation (A.12)). These scaling dimensions coincide with those obtained from non-Abelian bosonization [29, 41].

Turning now to the SU(3)-symmetry-breaking terms, we find that in the Abelian bosonization representation they take the following form:

$$\begin{aligned} \tilde{H}_1 = & \int \frac{\mathrm{d}x}{\pi^2} \left[-\frac{qa}{4}(\pi\Pi_a)^2 - \frac{qa}{2}(\partial_x\phi_a)^2 + \frac{qa}{12}(\partial_x\phi_b)^2\right] \\ & + \frac{2qa}{(2\pi a)^2} \int \mathrm{d}x \, \cos\sqrt{8}\phi_a + \frac{\sqrt{3}qa}{(2\pi)^2} \int \mathrm{d}x \, \partial_x\phi_b \end{aligned} \tag{3.20}$$

$$\tilde{H}_2 = \int \frac{dx}{\pi^2}\left[\frac{5ua}{2}(\partial_x\phi_a)^2 + \frac{ua}{6}(\partial_x\phi_b)^2\right] + \frac{2ua}{4(\pi a)^2}\int dx\,\cos(\sqrt{8}\phi_a) + \int dx\,\frac{\sqrt{3}u}{(2\pi)^2}\,\partial_x\phi_b \tag{3.21}$$

$$\begin{aligned}\tilde{H}_3 = \int \frac{dx}{\pi^2}\left[\frac{u'a}{6}(\partial_x\phi_a)^2 + \frac{5u'a}{2}(\partial_x\phi_b)^2\right] - \frac{2u'a}{3(2\pi a)^2}\int dx\;\cos(\sqrt{8}\phi_a)\\ - \frac{4u'a}{3(\pi a)^2}\int dx\;\cos\sqrt{2}\phi_a\cos\sqrt{6}\phi_b + \frac{u'\sqrt{2}}{3\pi^2}\int dx\;\partial_x\phi_b\end{aligned} \tag{3.22}$$

$$\tilde{H}_h = -(h-h_c)\int dx\;\frac{\sqrt{2}}{\pi}(\partial_x\phi_b).$$

The physical interpretation of the terms proportional to $\partial_x\phi_b$ is very simple. The bosonized Hamiltonian is derived under the assumption that the magnetization per triangle is close to $5/(6a)$. When the magnetization per triangle is *exactly* $5/(6a)$ the terms $\partial_x\phi_b$ do not appear in the Hamiltonian. Therefore, the presence in the Hamiltonian of such terms means that the magnetic field needed to impose a magnetization of $5/(6a)$ per triangle is renormalized away from its bare value. Also, since the Hamiltonian preserves the symmetry between + and − chiralities, it is invariant under the transformation $\pi_a \to -\pi_a$, $\phi_a \to -\phi_a$. In particular, this precludes the terms $\partial_x\phi_a$ from appearing in the Hamiltonian.

Assembling all terms, we finally have the following field theory describing the spin sector of the SU(3) Hubbard model in the presence of symmetry-breaking perturbations:

$$\begin{aligned}H = v_F\sum_{i=a,b}\int\frac{dx}{2\pi}\,[(\pi\Pi_i)^2 + (\partial_x\phi_i)^2] + + \frac{2g_1}{(2\pi a)^2}\int dx\,\cos\sqrt{8}\phi_a(x)\\ + \frac{4g_2}{(2\pi a)^2}\int dx\,\cos\sqrt{2}\phi_a(x)\cos\sqrt{6}\phi_b(x) + \frac{g_4}{\pi^2}\int dx\;(\partial_x\phi_a)^2\\ + \frac{g_5}{\pi^2}\int dx\;(\partial_x\phi_b)^2 + \frac{h}{\pi}\int dx\;\partial_x\phi_b\end{aligned} \tag{3.23}$$

with

$$v_F = 2ta\sin(k_F a) = \sqrt{3}ta$$
$$h = \left(\frac{\sqrt{3}}{4\pi}u + \frac{\sqrt{2}}{3\pi}u' + \frac{\sqrt{2}}{2\pi}q\right)$$

and t the hopping amplitude. In our units, $t = 1$.

The notation can be made more compact by introducing the vectors

$$\vec{\phi} = (\phi_a, \phi_b)$$

and

$$\vec{\alpha}_1 = (1,0) \qquad \vec{\alpha}_2 = (1/2, \sqrt{3}/2) \qquad \vec{\alpha}_3 = (1/2, -\sqrt{3}/2)$$

where

$$\begin{aligned}K_a &= \left[\left(1 - \frac{qa}{2\pi v_F}\right)\Big/\left(1 - \frac{Ua}{\pi v_F} - \frac{qa}{\pi v_F} + \frac{u'a}{3\pi v_F} + \frac{5ua}{\pi v_F}\right)\right]^{1/2}\\ u_a &= v_F\left[\left(1 - \frac{qa}{2\pi v_F}\right)\left(1 - \frac{Ua}{\pi v_F} - \frac{qa}{\pi v_F} + \frac{u'a}{3\pi v_F} + \frac{5ua}{\pi v_F}\right)\right]^{1/2}\end{aligned} \tag{3.24a}$$

$$K_b = \left(1 - \frac{Ua}{\pi v_F} + \frac{5u'a}{\pi v_F} + \frac{ua}{3\pi v_F} + \frac{qa}{6\pi v_F}\right)^{-1/2}$$
$$u_b = v_F\left(1 - \frac{Ua}{\pi v_F} + \frac{5u'a}{\pi v_F} + \frac{ua}{3\pi v_F} + \frac{qa}{6\pi v_F}\right)^{1/2}. \qquad (3.24b)$$

The Hamiltonian can then be rewritten† as

$$H = \int \frac{\mathrm{d}x}{2\pi}\left[u_a K_a (\pi \Pi_a)^2 + \frac{u_a}{K_a}(\partial_x \phi_a)^2 + u_b K_b (\pi \Pi_b)^2 + \frac{u_b}{K_b}(\partial_x \phi_b)^2\right] + \sum_{i=1}^{3} \frac{2g_i}{(2\pi a)^2}\int \mathrm{d}x \cos(\sqrt{8}\vec{\alpha}_i \cdot \vec{\phi})$$

The interactions $(\partial_x \phi_{a,b})^2$ can render the marginal operators $\cos(\sqrt{8}\phi_a)$, $\cos(\sqrt{2}\phi_a \pm \sqrt{6}\phi_b)$ marginally relevant and cause the opening of a gap. In such a case, the low-energy properties of the system cannot be described by two massless bosons. One can have either a massive and a massless boson or two massive bosons. This depends on the coupling constants u, u', m, q, and the magnetic field h. In order to explore this possibility in more detail, one has to use renormalization group equations. This is the subject of the forthcoming sections.

4. The renormalization group flow in zero magnetic field

In this section, we discuss the flow of the renormalization group equation and the phase diagram that results from it. Qualitatively, the renormalization group equations are similar to the Kosterlitz–Thouless renormalization group equations [44, 45]. We expect therefore to obtain a gapless phase corresponding to the flow to a fixed hypersurface of the six-dimensional space of coupling constants and one (or possibly many) gapped phase where the coupling constants flow to infinity. We also expect the phase transition to be of infinite order [44]. Our task is therefore to determine the initial conditions and follow the flow. This will allow us to conclude on the nature of the ground state of the anisotropic SU(3) chain.

A straightforward application to the Hamiltonian (3.23) of the standard method [45, 46] would be inconvenient since one needs to expand to third order of correlation functions in order to get the full one-loop RG equations [47]. We will use instead operator product expansion (OPE) techniques [48, 49]. In our case, the algebra of operators $(\partial_x \phi_a)^2$, $(\partial_x \phi_b)^2$, and $\cos(\sqrt{8}\vec{\alpha}_i \cdot \vec{\phi})$ closes under OPE (for details see appendix B). In particular,

$$\cos(\sqrt{8}\vec{\alpha} \cdot \vec{\phi}(x,\tau)) \cos(\sqrt{8}\vec{\alpha} \cdot \vec{\phi}(0,0)) \simeq \frac{-2a^4}{(x^2 + (u\tau)^2)^2}\left[\sum_{p=a,b} (\alpha_i^p)^2 (x^2 (\partial_x \phi_p)^2 + \tau^2 (\partial_\tau \phi_p)^2 + 2x\tau\, \partial_x \phi_p\, \partial_\tau \phi_p)\right]$$

$$\cos(\sqrt{8}\vec{\alpha} \cdot \vec{\phi}(x,\tau)) \cos(\sqrt{8}\vec{\beta} \cdot \vec{\phi}(0,0)) \simeq \frac{1}{2}\left[\left(\frac{a^2}{x^2 + (u\tau)^2}\right)^{-K\vec{\alpha}\cdot\vec{\beta}} \cos\sqrt{8}(\vec{\alpha} + \vec{\beta}) \cdot \vec{\phi}(0,0) + \left(\frac{a^2}{x^2 + (u\tau)^2}\right)^{K\vec{\alpha}\cdot\vec{\beta}} \cos\sqrt{8}(\vec{\alpha} - \vec{\beta}) \cdot \vec{\phi}(0,0)\right]$$

† It is interesting to note that a similar Hamiltonian enters in the theory of two-dimensional melting (see equation (3.16) in reference [47]).

lead to the following RG equations (see appendix B):

$$
\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}l} &= 2y_1y_4 - y_2^2/2 \\
\frac{\mathrm{d}y_2}{\mathrm{d}l} &= (\tfrac{1}{2}y_4 + \tfrac{3}{2}y_5)y_2 - y_1y_2/2 \\
\frac{\mathrm{d}y_4}{\mathrm{d}l} &= y_1^2/2 + y_2^2/4 \\
\frac{\mathrm{d}y_5}{\mathrm{d}l} &= \tfrac{3}{4}y_2^2
\end{aligned}
\tag{4.1}
$$

where we have used the notation $y_i = g_i/\pi v_F$, with v_F the Fermi velocity. Note that $(g_1, g_2) = (0, 0)$ is a fixed surface, because of three truly marginal operators, $\partial_x\phi_a\,\partial_x\phi_b$, $(\partial_x\phi_a)^2$, and $(\partial_x\phi_b)^2$.

An alternative approach based on non-Abelian bosonization can be used. In this approach, after expressing the Hamiltonian in terms of products of right- and left-moving currents $J_R^a J_L^a$, an operator product expansion for currents is derived [50]. Such an approach leads to the same RG equations as the Abelian bosonization approach.

The initial values of the running coupling constants (at the cut-off scale a) for the spin sector of the SU(3) Hubbard model perturbed by $\tilde{H}_{1,2,3}$ are given by

$$
\begin{aligned}
y_1(a) &= \frac{g_1(a)}{\pi v_F} = \left(Ua + qa - \frac{u'a}{3} + ua\right)\Big/\pi v_F \\
y_2(a) &= \frac{g_2(a)}{\pi v_F} = \left(Ua + \frac{u'a}{3}\right)\Big/\pi v_F \\
y_4(a) &= \frac{g_4(a)}{\pi v_F} = \left(-\frac{Ua}{2} + \frac{5ua}{2} + \frac{qa}{2} + \frac{u'a}{6}\right)\Big/\pi v_F \\
y_5(a) &= \frac{g_5(a)}{\pi v_F} = \left(-\frac{Ua}{2} + \frac{ua}{6} + \frac{5u'a}{2} + \frac{qa}{12}\right)\Big/\pi v_F.
\end{aligned}
\tag{4.2}
$$

In the expression for the initial coupling constants (4.2) we have $u = -J/4$, $q = -J/12$, $u' = -5J/36$, and we assume $J, U \ll t$. Hereafter, we choose $J = 4$ and put v_F equal to unity; thus the numerical starting values are given by

$$
\begin{aligned}
y_1 &= -0.365\,467 + y_0 \\
y_2 &= -0.058\,9463 + y_0 \\
y_4 &= -0.878\,299 - y_0/2 \\
y_5 &= -0.503\,9907 - y_0/2.
\end{aligned}
\tag{4.3}
$$

Here $y_0 \propto U$. These values are not small, so the one-loop RG equations are not valid. However, numerically solving these RG equations with initial conditions (4.3) shows that they flow to a fixed point on the surface $(g_1, g_2) = (0, 0)$ for any $y_0 \in [0, 1]$ (see figures 3 and 4). At this fixed point, one has a renormalized Hamiltonian with

$$
H = \int \frac{\mathrm{d}x}{2\pi}\left[u_a^*K_a^*(\pi\Pi_a)^2 + \frac{u_a^*}{K_a^*}(\partial_x\phi_a)^2 + u_b^*K_b^*(\pi\Pi_a)^2 + \frac{u_b^*}{K_b^*}(\partial_x\phi_a)^2\right]. \tag{4.4}
$$

This proves that certainly for weak coupling the long-distance properties of the system are described by a two-component Luttinger liquid. For strong coupling, i.e. in the case of the spin tube, this is only an indication that the two-component Luttinger liquid is possible. In order to give a definitive proof, one should prove that there is no singularity in the ground-state energy as couplings increase.

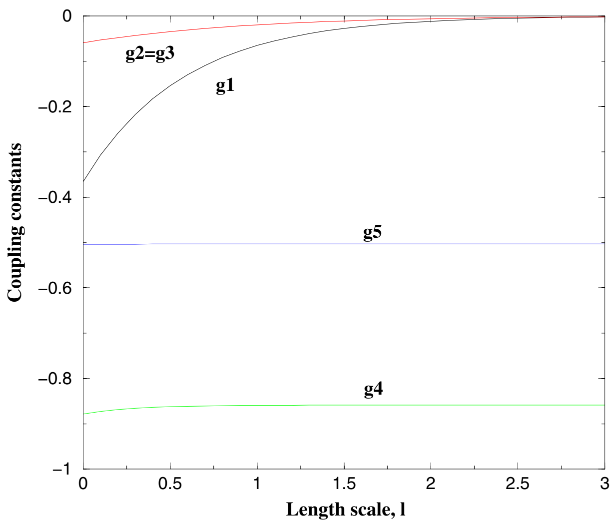

Figure 3. The renormalization group flow with initial conditions (4.3) and $y_0 = 0$. The coupling constants g_1, g_2, g_3 are renormalized to 0, whereas $g_4 \to g_4^*$, and $g_5 \to g_5^*$. The system therefore flows to a two-component Luttinger liquid fixed point.

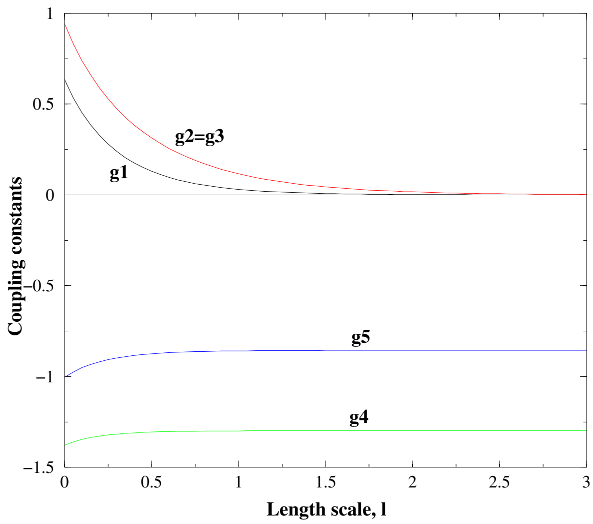

Figure 4. The renormalization group flow with initial conditions (4.3) and $y_0 = 1$. The coupling constants g_1, g_2, g_3 are renormalized to 0, whereas $g_4 \to g_4^*$, and $g_5 \to g_5^*$. The presence of a marginal perturbation preserving SU(3) symmetry does not suppress the two-component Luttinger liquid behaviour. However, by comparing with figure 3, it is seen that it changes the exponents at the fixed point.

Comparing figures 3 and 4, one can see that the magnitude of the fixed-point values of g_4 and g_5 depends on the strength of the marginal SU(3)-symmetric interaction y_0. This fact, combined with the fact that the RG equations are only valid for weak coupling precludes the use of the RG to give an accurate estimate of K_a^* and K_b^*. However, one can still determine from the RG equations whether these quantities are larger or smaller than one. Although we emphasize that these figures should not be given too much stress, we find, using the expressions for K_a and K_b as a function of g_4 and g_5, $K_a^* = 1.9$ and $K_b^* = 1.5$, i.e. both are larger than 1. Concerning the question of whether the two-component Luttinger liquid persists at large coupling, we can remark that the deviation from isotropy in our case makes the interaction between the SU(3) spins less antiferromagnetic. It is well known that in the case of the XXZ chain, reducing the antiferromagnetic character of the spin–spin interaction (i.e. working at $J_z < J$) prevents the formation of a gap [51]. Therefore, it seems likely that no gap would develop in the spectrum. To test our conjecture, numerical work, especially calculation of K_a^*, K_b^* by exact diagonalization, would prove very valuable.

The existence of a two-component Luttinger liquid phase has important consequences. In particular, it implies a non-zero magnetic susceptibility $\chi \propto K_b/u_b$, and a T-linear specific heat of the form

$$C = \frac{\pi T}{6u_a} + \frac{\pi T}{6u_b}. \tag{4.5}$$

The calculations of the correlation functions and NMR relaxation rate are deferred to section 5.

5. The strong-magnetic-field case: the fixed-point Hamiltonian and correlation functions

5.1. Generic magnetic field

5.1.1. Renormalization group flow under a magnetic field. Until now, we have not taken into account the terms associated with the magnetic field h_b, which can be treated as a perturbation, having fixed the external magnetic field at $h_c = (3/2)J_\perp$. To see whether the flow remains unchanged in this case, let us reobtain the renormalization group equations with finite h. The simplest way to address this problem is to perform a Legendre transformation [52] on the Hamiltonian (3.23). The non-zero average value of the field ϕ_b due to the finite magnetization can be eliminated by a simple shift of the ϕ_b-fields, i.e. $\phi_b = \phi_b - \pi m_b x$, where $m_b = -\langle \partial_x \phi_b \rangle / \pi$. One has the following relation between m_b and the magnetization:

$$m_b = -\sqrt{\frac{3}{2}}\left(\frac{\langle S^z \rangle}{a} - \frac{5}{6a}\right). \tag{5.1}$$

The cosine terms, however, are not invariant under this shift and the renormalization group equations (4.1) for the couplings g_1, g_2, and g_3, for a change of the length scale $a \to \mathrm{e}^l a$, now become (the details of the calculation can be found in appendix C)

$$\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}l} &= 2y_1y_4 - y_2^2 J_0(\pi m_b(l)a\sqrt{3})/2 \\
\frac{\mathrm{d}y_2}{\mathrm{d}l} &= \left(\tfrac{1}{2}y_4 + \tfrac{3}{2}y_5\right)y_2 - y_1y_2 J_0\left(\pi m_b(l)a\frac{\sqrt{3}}{2}\right)\Big/2 \\
\frac{\mathrm{d}y_4}{\mathrm{d}l} &= y_1^2/2 + y_2^2 J_0\left(\pi m_b(l)a\frac{\sqrt{3}}{2}\right)\Big/4 \\
\frac{\mathrm{d}y_5}{\mathrm{d}l} &= \tfrac{3}{4}y_2^2 J_0\left(\pi m_b(l)a\frac{\sqrt{3}}{2}\right)
\end{aligned} \tag{5.2}$$

where J_0 is the Bessel function that results from the use of a sharp cut-off in real space. One also has $m_b(l) = m_b(0)\mathrm{e}^l$. One can check that on setting $m_b(0) = 0$ one recovers equations (4.1). If the RG equation for the magnetization is trivial, the magnetic field, on the other hand, satisfies a non-trivial RG equation:

$$\frac{\mathrm{d}h_b}{\mathrm{d}l} = h_b + \frac{\sqrt{3}}{a\sqrt{8}} y_2^2 J_1(\pi\sqrt{6} m_b(l)a). \tag{5.3}$$

Let us discuss qualitatively the physics predicted by equations (5.2). One sees rather easily that for $m_b(l)a \ll 1$, the Bessel functions tend to zero, so one is left with a sine–Gordon renormalization group equation for y_1, y_4. Compared to the case of zero magnetization, we see that y_4 is more negative and y_1 is smaller in absolute value. Therefore, we expect that y_1 will be even more irrelevant in the presence of the finite magnetization. We conclude that the presence of a non-zero magnetization does not affect the two-component Luttinger liquid behaviour. The crossover scale can be roughly estimated as

$$l^\star \simeq \ln\left(\frac{v_F}{m_b a}\right). \tag{5.4}$$

At this crossover scale, the flow of y_5 is completely cut. This implies a variation of K_a, K_b with the magnetization.

At the value of l given by (5.4) the magnetic energy is of the order of energy cut-off; therefore the magnetic field term cannot be treated as a perturbation. When the initial magnetization goes to infinity the renormalization is stopped for smaller and smaller l. The coupling constants g_1, g_2, g_3 then become zero, while g_4, g_5 assume the values that they have at the scale l^*. Returning to the Hamiltonian (3.23), we see that it becomes a quadratic Hamiltonian.

5.1.2. The fixed-point Hamiltonian. Following the preceding discussion, we conclude that the asymptotic behaviour of the three-chain system under a magnetic field is governed by the Hamiltonian

$$H^\star = \int \frac{\mathrm{d}x}{2\pi}\, v_F \left[(\pi\Pi_a)^2 + (\partial_x\phi_a)^2 + (\pi\Pi_b)^2 + (\partial_x\tilde{\phi}_b)^2\right] + \frac{g_4^*}{\pi^2}\int \mathrm{d}x\, (\partial_x\phi_a)^2 + \frac{g_5^*}{\pi^2}\int \mathrm{d}x\, (\partial_x\phi_b)^2$$

where $g_{4,5}^*$ are functions of the magnetic field. The field $\tilde{\phi}_b$ is related to ϕ_b in the following way:

$$\phi_b = \tilde{\phi}_b + \pi\left(m - \frac{5}{6a}\right)\sqrt{\frac{3}{2}}x \tag{5.5}$$

while the dual fields θ_a and θ_b are not shifted. This condition guarantees that $\tilde{\phi}_b$ satisfies periodic boundary conditions.

The fixed-point Hamiltonian can be rewritten as

$$H^\star = \int \frac{\mathrm{d}x}{2\pi}\left[u_a^* K_a^* (\pi\Pi_a)^2 + \frac{u_a^*}{K_a^*}(\partial_x\phi_a)^2\right] + \int \frac{\mathrm{d}x}{2\pi}\left[u_b^* K_b^* (\pi\Pi_b)^2 + \frac{u_b^*}{K_b^*}(\partial_x\tilde{\phi}_b)^2\right] \tag{5.6}$$

where $u_i^* K_i^* = v_F; i = a, b$, and $u_{a,b}^*/K_{a,b}^* = v_F + 2g_{4,5}^*/\pi$. Both the velocities of the excitations, u_i, and the compactification radii, K_i, depend on the magnetic field h through $g_i^\star(h)$. Therefore, the low-energy properties of the system are described by two decoupled $c = 1$ conformal field theories with velocities and compactification radii depending on the applied magnetic field.

This is valid at the level of perturbation theory for the spin sector of the SU(3) Hubbard model. However, we are actually interested in the SU(3) anisotropic spin chain for which perturbation theory does not apply. In the latter case, we expect, relying on the continuity between the weak- and the strong-coupling regime, the anisotropic SU(3) spin chain under magnetic field to also be described by a two-component Luttinger liquid. However, the velocities and compactification radii cannot be obtained by perturbation theory techniques. Nevertheless, it is known that the velocities and compactification radii can be obtained by calculating only thermodynamic quantities using, for instance, exact-diagonalization techniques [39, 40]. The problem of the determination of these exponents in terms of measurable thermodynamic quantities in the specific case of the anisotropic SU(3) spin chain will be discussed in appendix D. The knowledge of the exponents then permits the calculation of the correlation functions. This is the subject of the next section.

5.1.3. Correlation functions. In this section, we want to calculate the three Matsubara correlation functions

$$\chi_{zz}(x,\tau) = \langle T_\tau S^z(x,\tau) S^z(0,0)\rangle \tag{5.7}$$

$$\chi_{+-,p}(x,\tau) = \langle T_\tau S^+_p(x,\tau) S^-_p(0,0)\rangle \tag{5.8}$$

$$\chi_{zz,p}(x,\tau) = \langle T_\tau S^z_p(x,\tau) S^z_p(0,0)\rangle \tag{5.9}$$

where $p = 1, 2, 3$ is a chain index. The first correlation function is useful for neutron scattering experiments, whereas the correlation functions (5.8) are useful for the calculation of NMR relaxation rates. The Matsubara correlation functions in Fourier space are given by

$$\chi_{ij}(q,\omega_n,T) = \int_0^\beta \mathrm{d}\tau\ \mathrm{d}x\ \mathrm{e}^{\mathrm{i}(\omega_n\tau - qx)} \langle T_\tau[S^i(x,\tau), S^j(0,0)]\rangle_T \tag{5.10}$$

from which the finite-temperature correlations are obtained by the analytic continuation $\mathrm{i}\omega_n \to \omega + \mathrm{i}0_+$. We will first concentrate on the $T = 0$ calculation, then explain how to extend the calculation to finite temperature.

We begin with the calculation of χ_{zz}. Using equation (2.9) we have

$$\chi_{zz} = \frac{1}{3}\langle T_\tau \Lambda^8(x,\tau)\Lambda^8(0,0)\rangle + (\langle S^z\rangle)^2. \tag{5.11}$$

Using the bosonized expressions for the SU(3) spins, equation (A.12), and the usual expression for the bosonized correlation functions [53], we obtain†

$$\begin{aligned}\chi_{zz} = (\langle S^z\rangle)^2 &+ \frac{K_b}{3\pi^2}\frac{(u_b\tau)^2 - x^2}{(x^2+(u_b\tau)^2)^2}\\ &+ \frac{\mathrm{e}^{\mathrm{i}(2\pi x/(3a) - \pi[m-5/(6a)])x}}{6(\pi a)^2}\left(\frac{a^2}{x^2+(u_a\tau)^2}\right)^{K_a/6}\left(\frac{a^2}{x^2+(u_b\tau)^2}\right)^{K_b/2}\\ &+ \frac{\mathrm{e}^{\mathrm{i}(2\pi x/(3a) + 2\pi[m-5/(6a)])x}}{3(\pi a)^2}\left(\frac{a^2}{x^2+(u_b\tau)^2}\right)^{2K_b/3} + \text{c.c.}\end{aligned} \tag{5.12}$$

where $m = \langle S^z\rangle/a$.

Turning to $\chi_{+-,p}$, it is easily seen using equation (2.8) that it is independent of p and equal to

$$\begin{aligned}\chi_{+-,p}(x,\tau) = \frac{1}{12}\big[&\langle T_\tau(\Lambda^1 + \mathrm{i}\Lambda^2)(x,\tau)(\Lambda^1 - \mathrm{i}\Lambda^2)(0,0)\rangle\\ &+ \langle T_\tau(\Lambda^4 + \mathrm{i}\Lambda^5)(x,\tau)(\Lambda^4 - \mathrm{i}\Lambda^5)(0,0)\rangle\big].\end{aligned}$$

† It is important to remark that the correlation functions have been calculated by using the fixed-point Hamiltonian. This means that there are logarithmic corrections to the expressions that we quote due to asymptotic freedom. Such logarithmic corrections have been analysed for instance in references [33, 46] for SU(2) spin chains.

Similarly, $\chi_{zz,p}$ is independent of p (see equation (2.8)) and the contribution not already included in χ_{zz} is of the form

$$\langle T_\tau(\Lambda^6 + \mathrm{i}\Lambda^7)(x,\tau)(\Lambda^6 - \mathrm{i}\Lambda^7)(0,0)\rangle. \tag{5.13}$$

The expressions for the required correlators are obtained as

$$\begin{aligned}
&\langle T_\tau(\Lambda^n + \mathrm{i}\Lambda^{n+1})(x,\tau)(\Lambda^n - \mathrm{i}\Lambda^{n+1})(0,0)\rangle \\
&\quad = \frac{2}{(\pi a)^2}\left[\left(\frac{a^2}{x^2+(u_a\tau)^2}\right)^{\nu_{n,1}}\left(\frac{a^2}{x^2+(u_b\tau)^2}\right)^{\nu_{n,2}}\cos(Q_n x + \Phi_n(\tau/x))\right. \\
&\qquad \left. + \left(\frac{a^2}{x^2+(u_a\tau)^2}\right)^{\eta_{n,1}}\left(\frac{a^2}{x^2+(u_b\tau)^2}\right)^{\eta_{n,2}}\cos\left(\frac{2\pi}{3a} + Q'_n x + \Psi_n(\tau/x)\right)\right]
\end{aligned} \tag{5.14}$$

where $n = 1, 4, 6$. The exponents are given by

$$\nu_{1,1} = \frac{1}{2K_a} + \frac{K_a}{2} \qquad \nu_{1,2} = 0 \tag{5.15}$$

$$\eta_{1,1} = \frac{1}{2K_a} \qquad \eta_{1,2} = \frac{K_b}{6} \tag{5.16}$$

$$\nu_{4,1} = \nu_{6,1} = \frac{1}{8K_a} + \frac{K_a}{8} \qquad \nu_{4,2} = \nu_{6,2} = \frac{3}{8K_b} + \frac{3K_b}{8} \tag{5.17}$$

$$\eta_{4,1} = \eta_{6,1} = \frac{1}{8K_a} + \frac{K_a}{8} \qquad \eta_{4,2} = \eta_{6,2} = \frac{3}{8K_b} + \frac{K_b}{24}. \tag{5.18}$$

It can be checked that for $u_a = u_b$, $K_a = K_b = 1$, one recovers the exponents of the isotropic SU(3) spin chain [29], namely $\nu_{n,1} + \nu_{n,2} = 1$ and $\eta_{n,1} + \eta_{n,2} = 2/3$. One also has

$$\begin{aligned}
&Q_1 = 0 && Q'_1 = -\pi\left(m - \frac{5}{6a}\right) \\
&Q_4 = \frac{3}{2}Q'_1 = -Q_6 && Q'_4 = -\frac{Q'_1}{2} = -Q'_6.
\end{aligned} \tag{5.19}$$

Recall that $\langle S^z\rangle = 5/6 + \sqrt{2/3}m_b$. This allows the determination of all incommensurate modes. Finally, we have the functions

$$\begin{aligned}
&\Phi_1\left(\frac{\tau}{x}\right) = 2\arctan\left(\frac{u_a\tau}{x}\right) \\
&\Psi_1\left(\frac{\tau}{x}\right) = 0 \\
&\Phi_4\left(\frac{\tau}{x}\right) = \Phi_6\left(\frac{\tau}{x}\right) = \frac{1}{2}\arctan\left(\frac{u_a\tau}{x}\right) + \frac{3}{2}\arctan\left(\frac{u_b\tau}{x}\right) \\
&\Psi_4\left(\frac{\tau}{x}\right) = -\Psi_6\left(\frac{\tau}{x}\right) = \frac{1}{2}\left[\arctan\left(\frac{u_a\tau}{x}\right) - \arctan\left(\frac{u_b\tau}{x}\right)\right].
\end{aligned} \tag{5.20}$$

All the preceding results are valid only at $T = 0$. However, it is useful also to calculate the correlation functions for $T > 0$, in particular in order to obtain NMR relaxation rates. To obtain the finite-temperature Matsubara correlation functions, we can use a conformal transformation since we have two decoupled $c = 1$ conformal field theories. The explicit expression for this transformation is

$$x + \mathrm{i}u_i\tau \to \beta u_i \sinh\left(\frac{2\pi(x + \mathrm{i}u_i\tau)}{\beta u_i}\right) \tag{5.21}$$

where $i = a, b$. Therefore, to obtain† the finite-temperature Matsubara correlation functions, one has to make the substitutions

$$
\begin{aligned}
x^2 + (u_i\tau)^2 &\to (\beta u_i)^2 \left[\cosh^2\left(\frac{2\pi x}{\beta u_i}\right) - \cos^2\left(\frac{2\pi\tau}{\beta}\right)\right] \\
\arctan\left(\frac{u_i\tau}{x}\right) &\to \arctan\left(\frac{\tan(2\pi\tau/\beta)}{\tanh(2\pi x/(\beta u_i))}\right).
\end{aligned}
\tag{5.22}
$$

With the help of the above results for the spin–spin correlation functions, we can evaluate the T-dependence of the NMR longitudinal relaxation rate T_1:

$$
\frac{1}{T_1} \propto \lim_{\omega_n\to 0}\int_0^\beta \mathrm{d}\tau\ \mathrm{e}^{\mathrm{i}\omega_n\tau}\langle T_\tau S_p^+(0,\tau)S_p^-(0,0)\rangle_T. \tag{5.23}
$$

We find

$$
\begin{aligned}
\frac{1}{T_1} \propto \big(&aT^{1/(2K_a)+K_a/2-1} + bT^{1/(2K_a)+K_b/6} + cT^{(3K_a+K_b)/24+1/(8K_a)+3/(8K_b)-1} \\
&+ dT^{1/(8K_a)+3/(8K_b)+(3K_b+K_a)/8-1}\big).
\end{aligned}
$$

The low-temperature exponent is the smallest of the four exponents above.

5.1.4. Comparison with a spin-1 chain with biquadratic coupling. In the case of a bilinear biquadratic spin-1 chain defined by the Hamiltonian

$$
H = J\sum_i \vec{S}_i\cdot\vec{S}_{i+1} + \beta(\vec{S}_i\cdot\vec{S}_{i+1})^2 \tag{5.24}
$$

close to the Uimin–Lai–Sutherland point ($\beta \simeq 1$), a mapping onto an anisotropic SU(3) spin chain is also possible [20]. However, there are important differences. First, the expression for the spin operators in terms of the Gell-Mann matrices is different from the ones obtained in the spin-tube case. For the spin-1 case, one has

$$
\begin{aligned}
S_n^x &= \frac{\sqrt{2}}{2}(\lambda_n^4 + \lambda_n^6) \\
S_n^y &= \frac{\sqrt{2}}{2}(\lambda_n^5 - \lambda_n^7) \\
S_n^z &= \lambda_n^3.
\end{aligned}
\tag{5.25}
$$

These expressions should be contrasted with equations (2.8) and (2.9).

Although expressions (5.25) lead to incommensurate modes, the expressions for the correlation functions are different from the case for the spin tube. Second, the expression for the Hamiltonian in terms of λ-matrices in the spin-1 case is different from expression (2.17). That is, the Hamiltonian (5.24) can be rewritten in terms of Gell-Mann matrices as

$$
\begin{aligned}
H = \sum_i\bigg[&\frac{\beta}{2}(\lambda_i^8\lambda_{i+1}^8 + \lambda_i^1\lambda_{i+1}^1 + \lambda_i^2\lambda_{i+1}^2) + \left(1-\frac{\beta}{2}\right)\lambda_i^3\lambda_{i+1}^3 \\
&+ \frac{1}{2}(\lambda_i^4\lambda_{i+1}^4 + \lambda_i^5\lambda_{i+1}^5 + \lambda_i^6\lambda_{i+1}^6 + \lambda_i^7\lambda_{i+1}^7) \\
&+ \frac{1-\beta}{2}(\lambda_i^4\lambda_{i+1}^6 + \lambda_i^6\lambda_{i+1}^4 - \lambda_i^5\lambda_{i+1}^7 - \lambda_i^7\lambda_{i+1}^5)\bigg].
\end{aligned}
\tag{5.26}
$$

† The conformal transformation was carried out right at the fixed point. For a system that does not sit exactly at the fixed point, there would be corrections coming from the irrelevant operators. This is the finite-temperature counterpart of the logarithmic corrections that are obtained at $T = 0$.

Finally, for $\beta < 1$ the spin-1 bilinear biquadratic chain has a gap and the two-component Luttinger liquid can only be observed for a large enough applied magnetic field.

Nevertheless, the two problems have in common the presence of a gapless two-component Luttinger liquid ground state [20], and the formation of incommensurate modes under a magnetic field, so loosely speaking they belong to the same universality class. This can be understood as a consequence of the fact that both models can be related to anisotropic SU(3) spin chains. One should note that the formation of incommensurate modes in the presence of the magnetic field in the spin tube is not related to the presence of *gapped* incommensurate modes in the bilinear–biquadratic spin-1 chain [54]. In the latter case, the incommensurate modes originate from the fact that in the absence of the biquadratic chain, the (gapped) modes of the spin-1 chain are at $q = 0$ and $q = \pi$, whereas at the ULS point, the (gapless) modes are at $q = 0$ and $q = 2\pi/3$. The presence of gapped incommensurate modes between these two limits is merely a consequence of the continuity of the transition between the Haldane gap phase and the gapless phase beyond the ULS point. On the other hand, in the presence of the magnetic field, the gapless modes of the spin tube or those of the spin-1 chain simply move away from $2\pi/3$, similarly to what happens in a single spin-1/2 chain.

5.2. Is there a magnetization plateau for $\langle S^z \rangle = 1$?

5.2.1. The Umklapp terms and the quantization condition on the magnetization. In the presence of a magnetic field, one of the central issues is the quantization condition on the total magnetization $\langle S^z \rangle$ for the appearance of plateaus. This condition may be investigated by looking at the bosonized expression for the spin operators (A.12). After using the transformation (A.7) to take into account a non-zero magnetization, one can rederive an expression for the non-SU(3)-symmetric perturbations. Contrary to the case for zero magnetization, we cannot assume *a priori* that the '$4k_F$', '$6k_F$', ... terms are highly oscillating in some exceptional cases, since the phase $e^{i2nk_F x}$ may be compensated by a phase arising from the transformation (A.7) at the special filling (5.5). A systematic investigation indicates that the possible Umklapp term is originating from the terms $\lambda_i^3 \lambda_{i+1}^3$, $\lambda_i^8 \lambda_{i+1}^8$ in the Hamiltonian (3.1) and is

$$\cos\left(4\sqrt{\frac{2}{3}}\tilde{\phi}_b + \alpha_3\right). \tag{5.27}$$

And if we consider the three Umklapp processes from a higher order in the strong-rung-coupling expansion, we have another operator:

$$\cos(2\sqrt{6}\tilde{\phi}_b). \tag{5.28}$$

The conditions that must be fulfilled for these Umklapp terms to be present are

$$\frac{\langle \Lambda^8 \rangle}{\sqrt{3}} = -\frac{1}{6} \text{ or } \frac{1}{3} \tag{5.29}$$

for the term (5.27) and

$$\langle \Lambda^8 \rangle = -\frac{n}{\sqrt{3}} \tag{5.30}$$

for the term (5.28).

Condition (5.29) reduces to $\langle S^z \rangle = 1$ and condition (5.30) reduces to $\langle S^z \rangle = 7/6$ or $\langle S^z \rangle = 5/6$. At these values of the magnetization the operators (respectively) (5.27) or (5.28) can in principle open a gap in the excitations of the system leading to a plateau in the magnetization curve. In agreement with the generalized LSM theorem [7–9], the formation of such a gap implies a degeneracy of the ground state. This degeneracy is usually associated with a breaking of translational symmetry in the ground state. Under the assumption of broken translational symmetry, for $\langle S^z \rangle = 1$, the ground state has period 2, and in the case of $\langle S^z \rangle = 5/6, 7/6$, it has period 3 [7–9]. For this to happen, at least for the magnetic excitation, these operators must be relevant to make the magnetization plateau; that is,

$$K_b < 3/4 \tag{5.31}$$

for the term (5.27) and

$$K_b < \frac{1}{3} \tag{5.32}$$

for the term (5.28). According to these equations the presence of a plateau is more likely at $\langle S^z \rangle = 1$ than at $\langle S^z \rangle = 7/6$. Also we should note that the non-Umklapp sine–Gordon operator

$$\cos\sqrt{2}\phi_a \cos\sqrt{6}\phi_b \tag{5.33}$$

can appear only at $\langle S^z \rangle = 5/6$. Let us remark that for $\langle S^z \rangle = 5/6$, the RG analysis of the previous section showed that no gap would result since $K_b > 1$.

5.2.2. A theory of the possible plateau at $\langle S^z \rangle = 1$. There seems to be evidence for an extra plateau at $\langle S^z \rangle = 1$ in the magnetization curve of a 36-site system at $J_\perp/J = 3$ obtained by DMRG techniques in reference [18] (see figure 10 of reference [18]). However, this small plateau observed at $\langle S^z \rangle = 1$ in the magnetization curve may also be ascribed to the small system size [55]. Here, we will investigate in more detail the possibility of such a plateau in the framework of the bosonized theory. In particular, we will try to give a description of the behaviour of correlation functions in the system.

Due to the presence of the Umklapp term (5.27) the bosonized Hamiltonian describing the low-energy excitations of the spin tube at $\langle S^z \rangle = 1$ is

$$\begin{aligned} H &= H_a + H_b \\ H_a &= \int \mathrm{d}x\, \frac{\mathrm{d}x}{2\pi}\left[u_a K_a (\pi \Pi_a)^2 + \frac{u_a}{K_a}(\partial_x \phi_a)^2\right] + \frac{2g_1}{(2\pi a)^2}\int \mathrm{d}x\ \cos\sqrt{8}\phi_a \\ H_b &= \int \mathrm{d}x\, \frac{\mathrm{d}x}{2\pi}\left[u_b K_b (\pi \Pi_b)^2 + \frac{u_b}{K_b}(\partial_x \phi_b)^2\right] + \frac{2u'a}{3(\pi a)^2}\int \mathrm{d}x\, \cos\left(4\sqrt{\frac{2}{3}}\phi_b + \frac{2\pi}{3}\right). \end{aligned} \tag{5.34}$$

If there is indeed a magnetization plateau at $\langle S^z \rangle = 1$, a gap opens in the excitation spectrum of the field ϕ_b. However, the Hamiltonian contains no terms coupling ϕ_a and ϕ_b, so ϕ_a could remain ungapped. This would lead to a single-component Luttinger liquid behaviour on this plateau, and a power-law decay of some spin–spin correlation functions. Since $u' < 0$, the Umklapp term would impose

$$\sqrt{\frac{2}{3}}\langle\tilde{\phi}_b\rangle = -\frac{2\pi}{3}.$$

If this is so, careful treatment of the expressions (A.12) of the bosonized forms of the Λ operators is necessary when we have a gap, on the $\langle S^z \rangle = 1$ plateau. To eliminate gapped states completely, one can use the following expression (see appendix A):

$$\Lambda_i^8 = \frac{1}{\sqrt{3}}(1 - 3c_{i,3}^\dagger c_{i,3}) \tag{5.35}$$

for S_i^z instead of (3.12). Here we use the constraint (3.11). This equation indicates that we have only the gapful excitations ϕ_b in $\Lambda^8(x)$. One has

$$\begin{aligned}(\Lambda^1 + \mathrm{i}\Lambda^2)(x) &= \frac{\mathrm{e}^{\mathrm{i}\sqrt{2}\theta_a}}{\pi a}\left[2\cos\sqrt{2}\phi_a + 2C_1\cos\left(\frac{\pi}{2a}x + \frac{\pi}{6}\right)\right] \\ \Lambda^8(x) &= \frac{3}{\pi a\sqrt{3}}\mathrm{e}^{\mathrm{i}(\pi/a)x}C_2\end{aligned} \tag{5.36}$$

where

$$\begin{aligned}C_1 &= \langle \mathrm{e}^{\mathrm{i}\sqrt{2/3}(\tilde{\phi}_b - \langle\tilde{\phi}_b\rangle)}\rangle \\ C_2 &= \langle \mathrm{e}^{\mathrm{i}\,2\sqrt{2/3}(\tilde{\phi}_b - \langle\tilde{\phi}_b\rangle)}\rangle.\end{aligned} \tag{5.37}$$

We do not give the expressions for the other operators since they show exponential decay except for Λ^3, which never enters the spin-correlation function. We find then that translational symmetry is broken on the $\langle S^z\rangle = 1$ plateau, with a period of 2 for the ground state.

This is in agreement with the LSM theorem that rules out a non-degenerate ground state if there is a magnetization plateau at $\langle S^z\rangle = 1$. The period that we obtain is in agreement with the LSM theorem if we assume that the degeneracy of the ground state results from a broken translational symmetry [7–9]. We also see that an additional period of 4 will appear in the correlation functions showing a power-law decay. The correlation function for the operators of equation (5.36) is

$$\begin{aligned}&\langle T_\tau(\Lambda^1 + \mathrm{i}\Lambda^2)(x,\tau)(\Lambda^1 - \mathrm{i}\Lambda^2)(x',0)\rangle \\ &\quad = \frac{1}{(\pi a)^2}\left[2C_1^2\left\{\left(1 + \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi x'/a}\right)\cos\left[\frac{\pi}{2a}(x - x')\right]\right.\right. \\ &\qquad \left. - \frac{\sqrt{3}}{2}\mathrm{e}^{\mathrm{i}\pi x'/a}\sin\left[\frac{\pi}{2a}(x - x')\right]\right\}\left(\frac{a^2}{(x - x')^2 + (u_a^*\tau)^2}\right)^{1/(2K_a)} \\ &\qquad \left. - \left(\frac{a^2}{(x - x')^2 + (u_a^*\tau)^2}\right)^{K_a/2 + 1/(2K_a)}\frac{(x - x')^2 - (u_a^*\tau)^2}{(x - x')^2 + (u_a^*\tau)^2}\right].\end{aligned}$$

Using expressions (2.8) and (2.9) for the spins in terms of λ-matrices, we find that the correlations $\langle S_n^+ S_{n'}^-\rangle$ show an exponential decay whereas the correlations $\langle S_{n,p}^z S_{n',p}^z\rangle$ follow a power-law decay. We have the following expressions for the equal-time spin–spin correlation functions:

$$\begin{aligned}\langle S_n^z\rangle &= 1 - \frac{C_2}{\pi}\mathrm{e}^{\mathrm{i}\pi n} \\ \langle S_{n,p}^z S_{n',p}^z\rangle - \langle S_{n,p}^z\rangle\langle S_{n',p}^z\rangle &= \frac{2}{9}\langle(\Lambda_n^1 + \mathrm{i}\Lambda_n^2)(\Lambda_{n'}^1 - \mathrm{i}\Lambda_{n'}^2)\rangle.\end{aligned} \tag{5.38}$$

Comparing with figures 6 and 7 of reference [18], one sees that such behaviour is not obtained in numerical calculations. This leaves two options: one is that the system size (36 sites) in reference [18] is too small for observing the finite but large correlation length. This is not unreasonable, since K_b could be only slightly smaller than 3/4. The second possibility is that the plateau at $\langle S^z\rangle = 1$ is an artifact of the small system size. In section 5.2.3 it will be shown using the DMRG for systems of up to 120 sites that it is the latter possibility that applies.

If we wish to obtain non-trivial plateaus, smaller values of Luttinger parameters K_a or K_b are needed. This could be achieved by adding a sufficiently strong antiferromagnetic Ising term along the chain:

$$\sum_{i=1}^{N}\sum_{p=1}^{3} S_{i,p}^z S_{i+1,p}^z \tag{5.39}$$

or an extra coupling:

$$\sum_{i=1}^{N}\sum_{p=1}^{3}\sum_{q=1}^{3}\vec{S}_{i,p}\cdot\vec{S}_{i+1,q}. \tag{5.40}$$

The extra plateaus would lie at $\langle S^z\rangle = 5/6, 1, 7/6$ and are allowed by an extended LSM theorem in the case of a degenerate ground state with broken translational symmetry.

5.2.3. Study of the presence of a magnetization plateau at $\langle S^z\rangle = 1$ by finite-size scaling of DMRG results. We continue the density-matrix renormalization group [13, 14] study of the three-leg ladder at $m = \langle S^z\rangle = 1$ of Tandon *et al* [18]. Their results for finite chain length show that there might be a plateau at $m = 1$ but the system size is too small to draw definitive conclusions [55]. In this section, we show that the apparent plateau at $m = 1$ is indeed a finite-size artifact. In finite-size study, there is a finite energy gap between any two non-degenerate energy levels. We therefore use $1/N$ scaling to show that the energy gap scales to zero in the thermodynamic limit for low-energy excitations with $\Delta S^{tot,z} = 0$ and $\Delta S^{tot,z} = 2$, respectively.

In our DMRG calculations we used periodic boundary conditions for the rung and open boundary conditions for the other direction—the i-direction in equation (2.1). With DMRG theory, we have gone up to 120 sites (chain length $N = 40$). The number of dominant density-matrix eigenstates, corresponding to the n largest eigenvalues of the density matrix, that we retained at each DMRG iteration was $n = 200$ with the biggest truncation error 10^{-6}. A few details of our DMRG procedure are worth mentioning here. For a system with length N, a given value of the magnetization per rung, m, corresponds to a sector with total spin S^z equal to $M = mN$. Using the infinite-system algorithm, we found the lowest energy states for $S^z = M-2$, $M-1$, M, $M+1$, $M+2$ and denoted the lowest energy in each S^z-sector as $E_0(S_z)$ and the second-lowest energy as $E_1(S_z)$. Since the total S^z is a good quantum number, it is more convenient to do numerical computations without including the magnetic field. Thus we have set $h = 0$ in equation (2.1) and used $J_\perp/J = 10$ to calculate $E_0(S_z)$ and $E_1(S_z)$. We have looked for a plateau at $m = 1$, and for each N we have calculated the $\Delta S^{tot,z} = 0$ gap as $E_1(S_z) - E_0(S_z)$, and the $\Delta S^{tot,z} = 2$ gap as $E_0(S_z-2) + E_0(S_z+2) - 2E_0(S_z)$. In figure 5 we have plotted $\Delta S^{tot,z} = 0$ versus $1/N$ and $\Delta S^{tot,z} = 2$ versus $1/N$ at magnetization $S_z = N$. The fittings to the second order of $1/N$ in the figure show that the gaps scale to zero in the forms $E_1(S_z) - E_0(S_z) \sim 1/N$ and $E_0(S_z+2) + E_0(S_z-2) - 2E_0(S_z) \sim 1/N$ when $N \to \infty$. Since there is no gap either in $\Delta S^{tot,z} = 0$ or in $\Delta S^{tot,z} = 2$ excitations, we deduce that the system is not dimerized, and that there is no plateau at $m = 1$. We can see the zero-gap excitations also by analysing the spectrum. Around $m = 1$ at length N and total spin S_z, we will have $3N/2 - S_z$ doublets given in equation (2.3). When $3N/2 - S_z$ is odd, the ground state is doubly degenerate due to the permutation symmetry of the two kinds of doublet, and the two ground states have parity $-$ with respect to the $i \to N+1-i$ reflection symmetry. When $3N/2 - S_z$ is even, the ground state is unique, but such unique ground states for N and for $N+4$ have different reflection parities. This suggests that there is no gap. There is no such parity change from N to $N+4$ for ground states of gapped translationally invariant systems or dimerized systems. This supports the basic picture in previous sections: when we increase or decrease S_z, we put in or take out gapless quasiparticles (doublets here); these gapless quasiparticles have different parities at different energy levels. These parities and degeneracies can be obtained by further detailed analysis of the low-energy excitations of the Luttinger liquid [56].

To obtain numerically the non-trivial plateaus, we need smaller values of the Luttinger parameters K_a or K_b. If we add a sufficiently strong Ising term along the chain direction (in J),

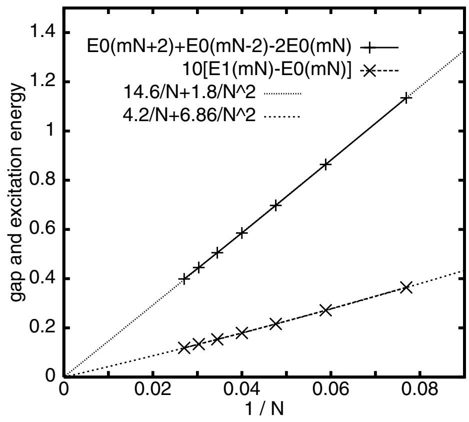

Figure 5. Gap scalings for periodic rung conditions at magnetization $m = 1$ and coupling $J_\perp/J = 10$. An XXX coupling along the chain direction is considered. The lowest energies $E_0(S_z)$ and $E_1(S_z)$ for each S_z obtained by DMRG techniques are plotted as a $\Delta S^{tot,z} = 0$ gap scaling: $10[E_1(S_z) - E_1(S_z)]$ versus $1/N$; and as a $\Delta S^{tot,z} = 2$ gap scaling: $E_0(S_z + 2) + E_0(S_z - 2) - 2E_0(S_z)$ versus $1/N$, with $S_z = mN$. We have magnified the $\Delta S^{tot,z} = 0$ gap by a factor of ten to make the figure clear. The linear fittings are $E_1(S_z) - E_0(S_z) \sim 1/N$ and $E_0(S_z+2)+E_0(S_z-2)-2E_0(S_z) \sim 1/N$ in the thermodynamic limit. The chain lengths calculated by DMRG techniques are $N = 12, 16, \ldots, 40$.

the condition is satisfied and we can obtain such plateaus at $\langle S^z \rangle = 5/6$, 1, and 7/6 as we predicted in previous subsections. These plateaus are allowed by an extended LSM theorem in the case of a periodic ground state with a certain periodicity, as explained in section 3. DMRG calculation for $\langle S^z \rangle = 5/6$ and 7/6 requires more technical effort [57]. In future studies, calculating numerically the Luttinger liquid exponents [58] would be useful to further the understanding of the three-legged ladder. In this section we study the magnetization plateau at $m = \langle S^z \rangle = 1$ for strong XXZ couplings in the chain direction. It bears out our predictions for the magnetization plateau at $m = 1$ made in previous sections.

In the following DMRG calculations we have used an XXZ coupling along the chain direction given by $J_z/J = 6$ and still used $J_\perp/J = 10$. In figure 6 we have plotted the $\Delta S^{tot,z} = 0$ and the $\Delta S^{tot,z} = 2$ gaps versus $1/N$ at $m = 1$ and for periodic rung conditions. The fittings to the second order of $1/N$ in the figure show that the $\Delta S^{tot,z} = 0$ gap scales to zero in the form $E_1(S_z) - E_0(S_z) \sim 1/N$ while the $\Delta S^{tot,z} = 2$ gap scales to a finite value when $N \to \infty$. So the magnetization plateau appears at $m = 1$ for periodic rung conditions *and* strong XXZ coupling in the chain direction. In order to make a comparison, we have also analysed the case of open rung conditions with the same J_z/J, $J_\perp/J$. For $m = 1$ we have plotted $E_1(S_z) - E_0(S_z)$ versus $1/N$ and $E_0(S_z - 2) + E_0(S_z + 2) - 2E_0(S_z)$ versus $1/N$ in figure 7. The fittings in the figure show that the $\Delta S^{tot,z} = 0$ gap scales to zero exponentially and the $\Delta S^{tot,z} = 2$ gap scales to a finite value when $N \to \infty$. So the magnetization plateau also appears for open rung conditions and strong XXZ coupling in the chain direction. It is thus not connected with the presence of frustrating boundary conditions, unlike the $m = 0$ plateau [9, 15, 18, 53].

6. Conclusions

In this paper, we have analysed the strong-coupling limit of the three-chain system with periodic boundary conditions in the presence of a magnetic field, using a mapping onto an anisotropic

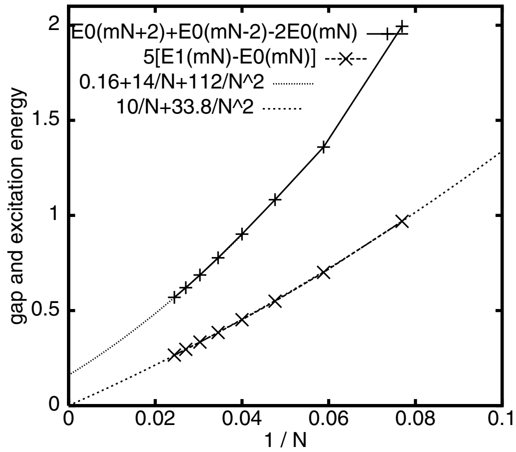

Figure 6. Gap scalings for periodic rung conditions at magnetization $m = 1$ for the coupling $J_\perp/J = 10$ and the XXZ coupling along the chain direction $J_z/J = 6$. The lowest energies $E_0(S_z)$ and $E_1(S_z)$ for each S_z obtained by DMRG techniques are plotted as a $\Delta S^{tot,z} = 0$ gap scaling: $5[E_1(S_z) - E_1(S_z)]$ versus $1/N$; and as a $\Delta S^{tot,z} = 2$ gap scaling: $E_0(S_z+2)+E_0(S_z-2)-2E_0(S_z)$ versus $1/N$, with $S_z = mN$. We have magnified the $\Delta S^{tot,z} = 0$ gap by a factor of five to make the figure clear. The linear fittings are $E_1(S_z) - E_0(S_z) \sim 1/N$ and $E_0(S_z+2)+E_0(S_z-2)-2E_0(S_z) \sim \Delta$ in the thermodynamic limit. The chain lengths calculated by DMRG techniques are $N = 12$, $16, \ldots, 40$.

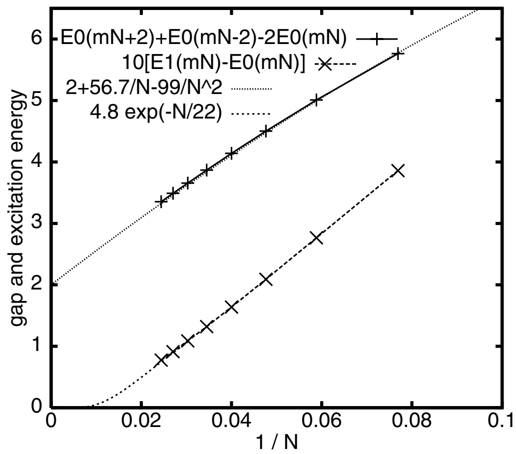

Figure 7. Gap scalings for open rung conditions at magnetization $m = 1$ for the coupling $J_\perp/J = 10$ and the XXZ coupling along the chain direction $J_z/J = 6$. The lowest energies $E_0(S_z)$ and $E_1(S_z)$ for each S_z obtained by DMRG techniques are plotted as a $\Delta S^{tot,z} = 0$ gap scaling: $10[E_1(S_z) - E_1(S_z)]$ versus $1/N$; and as a $\Delta S^{tot,z} = 2$ gap scaling: $E_0(S_z + 2) + E_0(S_z - 2) - 2E_0(S_z)$ versus $1/N$, with $S_z = mN$. We have magnified the $\Delta S^{tot,z} = 0$ gap by a factor of ten to make the figure clear. The fittings are $E_1(S_z) - E_0(S_z) \sim \exp(-N/\xi)$ and $E_0(S_z+2) + E_0(S_z-2) - 2E_0(S_z) \sim \Delta$ in the thermodynamic limit. The chain lengths calculated by DMRG techniques are $N = 12$, $16, \ldots, 40$.

SU(3) chain. A straightforward extension of the LSM theorem allowed us to locate the possible magnetization plateaus. Then, we applied bosonization and renormalization group techniques to show that for $1/2 < m < 3/2$, the system would be described by a two-component Luttinger liquid. This allowed us to obtain the spin–spin correlation functions of the system, and to follow the positions of the various incommensurate modes in the spin–spin correlation function as a function of the magnetization. Finally, we have predicted the temperature dependence of the NMR relaxation rate in this region. We also considered the evidence for a magnetization plateau at $m = 1$ in the framework of our bosonized description, and concluded that if such a plateau exists, the ground state at $m = 1$ should break translational symmetry with a period of two lattice spacings. We obtained the correlation functions of such a ground state as well as the average value of the magnetization at each site, and found that if such a plateau does exist the ground state would exhibit some kind of antiferromagnetic order. The numerical simulation of reference [18] shows no evidence for such antiferromagnetic order, thus pointing to an absence of any plateau at $m = 1$. To clarify whether or not there is a plateau at $m = 1$, we calculated the energy gap around $m = 1$ using the DMRG method and fitted the data to a linear function of the inverse system size. No gap was found in the thermodynamic limit; therefore, we conclude that there is no plateau at $m = 1$. To observe a non-trivial plateau at $m = 1$, we proposed modified ladder models. Adding a strong Ising term, we showed a plateau at this magnetization by DMRG techniques both for PBC and for OBC.

One obvious direction in which to extend our work is to calculate numerically the Luttinger liquid exponents of the three-chain ladder with periodic boundary conditions. In appendix D, indications can be found as to how these exponents could in principle be extracted. Finally, a generalization of the present analysis for N-odd cylindrically coupled $S = 1/2$ chains is important as well. Preliminary results show that a description of the low-energy effective Hamiltonian in terms of an anisotropic SU(3) spin chain still holds, expanding the result of a *two*-component Luttinger liquid to a more general context.

Acknowledgments

CI would like to thank I Affleck for kind hospitality extended to him at the University of British Columbia. EO acknowledges support from NSF under grant NSF DMR 96-14999. SQ acknowledges support from NSERC of Canada. RC acknowledges financial support from Università degli Studi di Salerno. The various authors are grateful to I Affleck, R Chitra, M Oshikawa and Paul Zinn-Justin for many useful and enlightening discussions. EO thanks D Sen for e-mail correspondence on the plateaus problem.

Appendix A. Abelian bosonization of the SU(3) Hubbard model

In this section, we apply Abelian bosonization to the SU(3) Hubbard model:

$$H_h = -t \sum_{i,n} [c^\dagger_{i,n} c_{i+1,n} + \text{h.c.}] + \frac{U}{2} \sum_{i,n\neq m} n_{i,n} n_{i,m}. \tag{A.1}$$

In the strong-coupling limit, the Hubbard Hamiltonian with one fermion per site projected onto the low-energy states becomes simply the Heisenberg Hamiltonian, as can be seen [41] by considering perturbation theory in the hopping term for $U \gg t$. Only second-order perturbation theory survives and the effective Heisenberg coupling is $J = t^2/U$.

In the continuum limit, in terms of the right- and left-moving fermions introduced in section 3, the free Hamiltonian H_t can be rewritten as

$$H_t = -\mathrm{i}v \int \mathrm{d}x \sum_n (\psi^\dagger_{R,n}\, \partial_x \psi_{R,n} - \psi^\dagger_{L,n}\, \partial_x \psi_{Ln}) \tag{A.2}$$

where $v = 2ta \sin(k_F a)$ is the Fermi velocity. In the following, we are working at a filling of one fermion per site. This implies $k_F = \pi/(3a)$ and $v_F = \sqrt{3}ta$.

Using the standard language of Abelian bosonization [36] we express $\psi_{R(L)n}$ in terms of the Bose fields ϕ_n and their duals θ_n, for each flavour $n = 1, 2, 3$:

$$\begin{aligned}\psi_{Rn}(x) &= \frac{1}{\sqrt{2\pi a}}\mathrm{e}^{\mathrm{i}(\theta_n(x)+\phi_n(x))}\eta_{Rn}\\ \psi_{Ln}(x) &= \frac{1}{\sqrt{2\pi a}}\mathrm{e}^{\mathrm{i}(\theta_n(x)-\phi_n(x))}\eta_{Ln}\end{aligned} \tag{A.3}$$

where $\eta_{R(L)n}$ are the Klein factors ensuring the proper anticommutation relations among fermion operators [53]. One has: $\pi\Pi_n(x) = \partial_x\theta_n$ and $[\phi_n(x), \Pi_m(x')] = \mathrm{i}\delta_{n,m}\delta(x - x')$.

The non-interacting Hamiltonian is straightforwardly rewritten as

$$H_t = \sum_{n=1}^{3} v \int \frac{\mathrm{d}x}{2\pi}\,\left[(\pi\Pi_n)^2 + (\partial_x\phi_n)^2\right] \tag{A.4}$$

and the fermion densities as

$$\rho_n(x) = -\frac{\partial_x\phi_n}{\pi} + \frac{\mathrm{e}^{-2\mathrm{i}k_F x}}{2\pi a}\mathrm{e}^{2\mathrm{i}\phi_n} + \frac{\mathrm{e}^{2\mathrm{i}k_F x}}{2\pi a}\mathrm{e}^{-2\mathrm{i}\phi_n} \tag{A.5}$$

where $k_F = \pi/(3a)$.

The Hubbard interaction

$$V = (Ua/2)\sum_{n\neq m}\int \mathrm{d}x\;\rho_n(x)\rho_m(x)$$

is rewritten in terms of the fields ϕ_n:

$$V = \int \mathrm{d}x \sum_{n\neq m}\frac{U}{\pi^2}\,\partial_x\phi_m\,\partial_x\phi_m + \frac{2U}{(2\pi a)^2}\cos 2(\phi_n - \phi_m). \tag{A.6}$$

Instead of working with fields ϕ_1, ϕ_2, ϕ_3 it is convenient [41] to introduce the transformation

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix}1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6}\\ 1/\sqrt{3} & 0 & -2/\sqrt{6}\end{pmatrix}\begin{pmatrix}\phi_c\\ \phi_a\\ \phi_b\end{pmatrix} \tag{A.7}$$

and similarly for the conjugate fields. The field ϕ_c describes the charge excitations, whereas the fields $\phi_{a,b}$ describe the SU(3) spin excitations. We recover in particular the fact that the SU(3) spin excitations are described by a conformal field theory with $C = 2$ whereas the charge excitations have $C = 1$. The charge and spin sectors of the Hubbard Hamiltonian are then completely separated:

$$H = H_c + H_s \tag{A.8}$$

$$H_c = \int \frac{\mathrm{d}x}{2\pi}\left[u_c K_c(\pi\Pi_c)^2 + \frac{u_c}{K_c}(\partial_x\phi_c)^2\right] \tag{A.9}$$

$$\begin{aligned}H_s = &\sum_{i=a,b}\int \frac{\mathrm{d}x}{2\pi}\left[u_s K_s(\pi\Pi_i)^2 + \frac{u_s}{K_s}(\partial_x\phi_i)^2\right]\\ &+ \frac{2g}{(2\pi a)^2}\int \mathrm{d}x\,\left[\cos(\sqrt{8}\phi_a) + \cos\sqrt{2}(\phi_a + \sqrt{3}\phi_b) + \cos\sqrt{2}(\phi_a - \sqrt{3}\phi_b)\right]\end{aligned} \tag{A.10}$$

where

$$\begin{aligned}&u_c K_c = v_F \qquad \frac{u_c}{K_c} = v_F + \frac{2Ua}{\pi}\\ &u_s K_s = v_F \qquad \frac{u_s}{K_s} = v_F - \frac{Ua}{\pi} \qquad g = Ua.\end{aligned} \tag{A.11}$$

Note that the Hamiltonian describing the charge modes contains *no* Umklapp term that could lead to a gap opening. This is due to the fact that equation (A.5) has been truncated at the $2k_F$ harmonic. In a more complete expression [59], higher harmonics would appear and would give a higher-order $6k_F$ Umklapp term. This Umklapp term can also be derived by perturbation theory [41]. This Umklapp term is of the form $\cos 2\sqrt{3}\phi_c$ and is *irrelevant* for $U/t \ll 1$. This is confirmed by numerical simulations [41] which show that the charge gap in an SU(3) Hubbard model open only for $U > 2.2t$. When considering RG equations (see section 3) we shall see that in the spin sector, for U initially positive, K_s renormalizes to 1 and g renormalizes to 0. This implies that the spin sector of the SU(3) Hubbard is described by a $C = 2$ conformal field theory perturbed by a marginally irrelevant operator [29, 41].

Of course, we also need a bosonized expression for the SU(3) spin operators. This can be derived from the continuum limit of the definition (3.12) of these operators; recall that $\Lambda^\alpha(x) \simeq a^{-1}\Lambda_i^\alpha$, $x = ia$ ($\alpha = 1, \ldots, 8$). We obtain

$$\begin{aligned}
\Lambda^1(x) &= \frac{\cos\sqrt{2}\theta_a}{\pi a}\left[2\cos\sqrt{2}\phi_a + \mathrm{e}^{\mathrm{i}[2\pi/(3a)]x}\mathrm{e}^{-2\mathrm{i}\phi_b/\sqrt{6}} + \mathrm{e}^{-\mathrm{i}[2\pi/(3a)]x}\mathrm{e}^{2\mathrm{i}\phi_b/\sqrt{6}}\right]\\
\Lambda^2(x) &= \frac{\sin\sqrt{2}\theta_a}{\pi a}\left[2\cos\sqrt{2}\phi_a + \mathrm{e}^{\mathrm{i}[2\pi/(3a)]x}\mathrm{e}^{-2\mathrm{i}\phi_b/\sqrt{6}} + \mathrm{e}^{-\mathrm{i}[2\pi/(3a)]x}\mathrm{e}^{2\mathrm{i}\phi_b/\sqrt{6}}\right]\\
\Lambda^3(x) &= -\frac{\sqrt{2}}{\pi}\,\partial_x\phi_a + \left[\frac{\mathrm{i}}{\pi a}\mathrm{e}^{\mathrm{i}[2\pi/(3a)]x}\mathrm{e}^{-2\mathrm{i}\phi_b/\sqrt{6}}\sin\sqrt{2}\phi_a + \mathrm{h.c.}\right]\\
\Lambda^4(x) &= \frac{1}{\pi a}\cos\left(\frac{\theta_a}{\sqrt{2}} + \sqrt{\frac{3}{2}}\theta_b\right)\\
&\qquad\times\left[2\cos\left(\phi_a/\sqrt{2} + \sqrt{\frac{3}{2}}\phi_b\right) + \mathrm{e}^{\mathrm{i}[2\pi/(3a)]x}\mathrm{e}^{\mathrm{i}(\phi_a/\sqrt{2}-\phi_b/\sqrt{6})} + \mathrm{h.c.}\right]\\
\Lambda^5(x) &= \frac{1}{\pi a}\sin\left(\frac{\theta_a}{\sqrt{2}} + \sqrt{\frac{3}{2}}\theta_b\right)\\
&\qquad\times\left[2\cos\left(\phi_a/\sqrt{2} + \sqrt{\frac{3}{2}}\phi_b\right) + \mathrm{e}^{\mathrm{i}[2\pi/(3a)]x}\mathrm{e}^{\mathrm{i}(\phi_a/\sqrt{2}-\phi_b/\sqrt{6})} + \mathrm{h.c.}\right]\\
\Lambda^6(x) &= \frac{1}{\pi a}\cos\left(\frac{\theta_a}{\sqrt{2}} - \sqrt{\frac{3}{2}}\theta_b\right)\\
&\qquad\times\left[2\cos\left(\phi_a/\sqrt{2} - \sqrt{\frac{3}{2}}\phi_b\right) + \mathrm{e}^{\mathrm{i}[2\pi/(3a)]x}\mathrm{e}^{\mathrm{i}(\phi_a/\sqrt{2}+\phi_b/\sqrt{6})} + \mathrm{h.c.}\right]\\
\Lambda^7(x) &= \frac{1}{\pi a}\sin\left(\frac{\theta_a}{\sqrt{2}} - \sqrt{\frac{3}{2}}\theta_b\right)\\
&\qquad\times\left[2\cos\left(\phi_a/\sqrt{2} - \sqrt{\frac{3}{2}}\phi_b\right) + \mathrm{e}^{\mathrm{i}[2\pi/(3a)]x}\mathrm{e}^{\mathrm{i}(\phi_a/\sqrt{2}+\phi_b/\sqrt{6})} + \mathrm{h.c.}\right]\\
\Lambda^8(x) &= -\frac{\sqrt{2}}{\pi}\,\partial_x\phi_b + \frac{\mathrm{e}^{\mathrm{i}[2\pi/(3a)]x}}{\pi\sqrt{3}a}\left[\mathrm{e}^{-\mathrm{i}\sqrt{2/3}\phi_b}\cos\sqrt{2}\phi_a - \mathrm{e}^{\mathrm{i}\sqrt{8/3}\phi_b}\right] + \mathrm{h.c.}
\end{aligned} \tag{A.12}$$

where $\Lambda^\alpha(x) = \Lambda_n^\alpha/a$ for $x = na$. Using these expressions, one can derive immediately the expressions (3.20)–(3.22). In the limit $U \to \infty$, one must note that the expression for $\Lambda_8(x)$ has to be modified. The reason for this is the following: for $U \to \infty$, one has $c_1^\dagger c_1 + c_2^\dagger c_2 + c_3^\dagger c_3 = 1$ for each site. As a result, $\lambda_8 = (c_1^\dagger c_1 + c_2^\dagger c_2 - 2c_3^\dagger c_3)/\sqrt{3}$ can be rewritten as $\lambda_8 = (1 - 3c_3^\dagger c_3)/\sqrt{3}$. Using bosonized expressions, one obtains

$$\Lambda_8(x) = -\frac{\sqrt{2}}{\pi}\,\partial_x\phi_b - \frac{\sqrt{3}}{2\pi a}\left[\mathrm{e}^{\mathrm{i}[2\pi/(3a)]x}\mathrm{e}^{\mathrm{i}\sqrt{8/3}\phi_b} + \mathrm{h.c.}\right]. \tag{A.13}$$

Thus, the terms containing $\cos\sqrt{2}\phi_a$ drop out of the expression for $\Lambda_8(x)$ in the limit $U \to \infty$. This means that the SU(3) Hubbard model with a finite charge gap and the SU(3) spin chain should have in general different correlations for Λ_8. It should be noted that this difference should not appear at the isotropic point, where the exponents of the correlation functions are identical. However, it is obtained for models in which the SU(3) rotation symmetry is broken.

Appendix B. Operator product expansion of marginal operators

In this section, our aim is to derive the OPE for operators of the form $\cos(\sqrt{8}\vec{\alpha}\cdot\vec{\phi})$, and deduce the renormalization group equations.

Let us first recall briefly how operator product expansions can be used to obtain one-loop renormalization group expansions [60]. Assume that we are given a set of operators Φ_k, closed under the OPE

$$\Phi_i(x,\tau)\Phi_j(0) \sim \sum_k c_{ij}^k(x,\tau)\Phi_k(0) \tag{B.1}$$

in the sense that expression (B.1), when inserted in any correlation function, gives the correct leading asymptotics for $(x,\tau) \to 0$. Denoting as $[\Phi_i]$ the scaling dimension of the operator Φ_i, defined by

$$\langle \Phi_i(x,0)\Phi_i(0,0)\rangle \propto \left(\frac{1}{x}\right)^{2[\Phi_i]} \tag{B.2}$$

we have $c_{ij}^k(x,0) \sim \text{constant} \times x^{[\Phi_k]-[\Phi_i]-[\Phi_j]}$.

Perturbing the Hamiltonian by

$$H_I = \sum_k \int \mathrm{d}x\ \mathrm{d}\tau\ g_k\Phi_k(x,\tau) \tag{B.3}$$

one can deduce the one-loop renormalization group beta functions directly from the OPE (B.1). These one-loop renormalization group equations are

$$\frac{\mathrm{d}g_k}{\mathrm{d}l} \equiv \beta_k(g) = (2-[\Phi_k])g_k - \pi\sum_{i,j} C_{ij}^k g_i g_j \tag{B.4}$$

where we define

$$C_{ij}^k = a^2\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\, c_{ij}^k\left(a\cos\theta, \frac{a}{u}\sin\theta\right). \tag{B.5}$$

The equations (B.4) are slight generalizations of those that can be found in reference [60], in which we have allowed for a function $c_{ijk}(x,\tau)$ that depends both on $x^2+u^2\tau^2$ and $u\tau/x$. The set of operators Φ_k has to be closed under the operator product expansion (i.e. they have to form a closed algebra), otherwise new operators would be generated under the RG, and new OPEs would have to be derived. We thus need to retain the smallest closed algebra that contains all the operators that appear in our problem. In our case, we have to retain the operators $\cos\sqrt{8}\vec{\alpha}_i\cdot\vec{\phi}$ as well as the operators $(\partial_x\phi_{a,b})^2$.

In order to derive the OPE for the $\cos\sqrt{8}\vec{\alpha}_i\cdot\vec{\phi}$ operators, we use the following identity:

$$\mathrm{e}^{\mathrm{i}\vec{\alpha}\cdot\vec{\phi}} = \mathrm{e}^{-\langle(\vec{\alpha}\cdot\vec{\phi})^2\rangle/2}{:}\mathrm{e}^{\mathrm{i}\vec{\alpha}\cdot\vec{\phi}}{:} \tag{B.6}$$

where $:\cdots:$ represents normal ordering. This identity implies

$$\mathrm{e}^{\mathrm{i}\sqrt{8}\vec{\alpha}\cdot\vec{\phi}(x,\tau)}\mathrm{e}^{-\mathrm{i}\sqrt{8}\vec{\alpha}\cdot\vec{\phi}(0,0)} = \mathrm{e}^{-4\langle(\vec{\alpha}\cdot[\vec{\phi}(x,\tau)-\vec{\phi}(0,0)])^2\rangle}{:}\mathrm{e}^{\mathrm{i}\sqrt{8}\vec{\alpha}\cdot[\vec{\phi}(x,\tau)-\vec{\phi}(0,0)]}{:}. \tag{B.7}$$

We have

$$\mathrm{e}^{-4\langle(\vec{\alpha}\cdot[\vec{\phi}(x,\tau)-\vec{\phi}(0,0)])^2\rangle} = \left(\frac{a^2}{x^2+(u\tau)^2}\right)^2 K. \tag{B.8}$$

And we can expand the normal-ordered product, yielding

$$:\mathrm{e}^{\mathrm{i}\sqrt{8}\vec{\alpha}\cdot[\vec{\phi}(x,\tau)-\vec{\phi}(0,0)]}: = 1 - 4(\vec{\alpha}\cdot[x\,\partial_x\vec{\phi}(0,0) + \tau\,\partial_\tau\vec{\phi}(0,0)])^2. \tag{B.9}$$

This leads to the OPE (4.1).

Now consider

$$\mathrm{e}^{\mathrm{i}\sqrt{8}\vec{\alpha}\cdot\vec{\phi}(x,\tau)}\mathrm{e}^{\mathrm{i}\sqrt{8}\vec{\beta}\cdot\vec{\phi}(0,0)} = \mathrm{e}^{-4\langle(\vec{\alpha}\cdot\vec{\phi}(x,\tau)+\vec{\beta}\cdot\vec{\phi}(0,0))^2\rangle} :\mathrm{e}^{\mathrm{i}\sqrt{8}(\vec{\alpha}\cdot\vec{\phi}(x,\tau)+\vec{\beta}\cdot\vec{\phi}(0,0))}:. \tag{B.10}$$

Rewriting this as

$$\langle(\vec{\alpha}\cdot\vec{\phi}(x,\tau)+\vec{\beta}\cdot\vec{\phi}(0,0))^2\rangle = \langle((\vec{\alpha}+\vec{\beta})\cdot\vec{\phi})^2\rangle - 2\vec{\alpha}\cdot\vec{\beta}(\langle\phi^2(0,0)-\phi(x,\tau)\phi(0,0)\rangle) \tag{B.11}$$

we have

$$\mathrm{e}^{\mathrm{i}\sqrt{8}\vec{\alpha}\cdot\vec{\phi}(x,\tau)}\mathrm{e}^{\mathrm{i}\sqrt{8}\vec{\beta}\cdot\vec{\phi}(0,0)} = \left(\frac{a}{\sqrt{x^2+u^2\tau^2}}\right)^{-2K\vec{\alpha}\cdot\vec{\beta}} \mathrm{e}^{\mathrm{i}\sqrt{8}(\vec{\alpha}+\vec{\beta})\cdot\vec{\phi}}. \tag{B.12}$$

The OPE for the cosines can be obtained trivially. There is no need to calculate the OPEs of the operators $(\partial_x\phi_{a,b})^2$ with the operators $\cos\sqrt{8}\vec{\alpha}_i\cdot\vec{\phi}$. It can be shown that these OPEs only reflect the dependence of the scaling dimensions of the cosines on $K_{a,b}$. Therefore, we have obtained all the OPEs needed to derive the renormalization group equations. It is then a simple matter to write the renormalization group equation using the formula (B.4). Following the procedure described in reference [60], one obtains

$$\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}l} &= (2-2K_a)y_1 - \frac{y_2y_3}{2\pi^2} \\
\frac{\mathrm{d}y_2}{\mathrm{d}l} &= \left(2-\frac{K_a}{2}-3\frac{K_b}{2}\right)y_2 - \frac{y_1y_3}{2} \\
\frac{\mathrm{d}y_3}{\mathrm{d}l} &= \left(2-\frac{K_a}{2}-3\frac{K_b}{2}\right)y_3 - \frac{y_1y_2}{2} \\
\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{1}{K_a}\right) &= \frac{y_1^2}{2}+\frac{y_2^2}{8}+\frac{y_3^2}{8} \\
\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{1}{K_b}\right) &= 3\frac{y_2^2}{8}+3\frac{y_3^2}{8}
\end{aligned} \tag{B.13}$$

where $y_i = g_i/(\pi v_F)$. A few remarks on these equations have to be made. In reference [60], the OPE depends only on the distance between points. In our case, the OPEs also depend on the angle between the segment joining the points and the horizontal axis. Since in the derivation of the RG equations one integrates over the ring $a < r < a\mathrm{e}^{\mathrm{d}l}$, the angular part of the integration cancels the terms $\partial_x\phi\,\partial_\tau\phi$ and gives a $\pi/2$ factor for the terms $(\partial_{x,\tau}\phi)^2$. The second important remark is that in our equations, we are working with $y_2(0) = y_3(0)$. It can be checked that this condition is preserved by the RG flow and that under such condition no terms $\partial_x\phi_a\,\partial_x\phi_b$ are generated. Finally, if we expand for small y_4, y_5, we have $K_a = 1 - y_4$ and $K_b = 1 - y_5$. Putting this in equations (B.13) we get the renormalization group equations.

Appendix C. Renormalization group equations in the presence of the external magnetic field

In the present section, we want to extend the derivation of the renormalization group equations of appendix B to the case of a non-zero effective magnetic field. As explained in section 5, it is convenient to first perform a Legendre transformation and work at fixed magnetization. Then, the RG equation for the magnetization becomes trivial, but the RG equation for the magnetic field is not. Similarly to the zero-magnetic-field case, the renormalization group equations can be obtained via operator product expansion. The only difficulty is that the problem is not *a priori* translationally invariant.

The relevant operator product expansions are obtained by the method of appendix B. One has

$$\begin{aligned}
&\cos(\sqrt{8}\vec{\alpha}\cdot[\vec{\phi}(x,\tau)+\vec{u}x])\cos(\sqrt{8}\vec{\alpha}\cdot[\vec{\phi}(x',\tau)+\vec{u}x']) \\
&\quad = \frac{a^4}{(x^2+u^2\tau^2)^2}\{-\sqrt{2}\sin(\sqrt{8}(\vec{u}\cdot\vec{\alpha})x)(x\,\partial_x(\vec{\alpha}\cdot\vec{\phi}) - \tau\,\partial_\tau(\vec{\alpha}\cdot\vec{\phi})) \\
&\quad - 2[(\vec{\alpha}\cdot(x\,\partial_x\vec{\phi}+\tau\,\partial_\tau\vec{\phi}))]^2\cos(\sqrt{8}(\vec{u}\cdot\vec{\alpha})x)\}.
\end{aligned} \tag{C.1}$$

In our case, one must take $\vec{u} = -\pi m_b(0,1)$. The term in $\tau\,\partial_\tau(\vec{\alpha}\cdot\vec{\phi})$ disappears upon angular integration. On the other hand, the term in $x\,\partial_x(\vec{\alpha}\cdot\vec{\phi})$ leads to a renormalization of the applied magnetic field. The angular integrations lead in general to Bessel functions.

The second useful OPE is

$$\begin{aligned}
&\mathrm{e}^{\mathrm{i}\sqrt{8}\vec{\alpha}\cdot(\vec{\phi}(x,\tau)+\vec{u}x)}\mathrm{e}^{\mathrm{i}\sqrt{8}\vec{\beta}\cdot(\vec{\phi}(x',0)+\vec{u}x')} \\
&\quad = \left(\frac{a}{\sqrt{(x-x')^2+u^2(\tau-\tau')^2}}\right)^{-2K\vec{\alpha}\cdot\vec{\beta}} \\
&\quad \times \mathrm{e}^{\mathrm{i}\sqrt{8}(\vec{\alpha}+\vec{\beta})\cdot(\vec{\phi}([x+x']/2,0)+\vec{u}[x+x']/2)}\mathrm{e}^{\mathrm{i}(\vec{\alpha}-\vec{\beta})\cdot\vec{u}[x-x']/2}.
\end{aligned} \tag{C.2}$$

These OPEs allow us to deduce the renormalization group equations for K_a, K_b, y_1, y_2, h in the form

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{1}{K_a}\right) &= \frac{1}{2}y_1^2 + \frac{1}{4}y_2^2 J_0\left(\pi m_b(l)\frac{\sqrt{3}}{2}a\right) \\
\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{1}{K_b}\right) &= \frac{3}{4}y_2^2 J_0\left(\pi m_b\sqrt{\frac{3}{2}}a(l)\right) \\
\frac{\mathrm{d}y_1}{\mathrm{d}l} &= (2-2K_a)y_1 - \frac{1}{2}y_2^2 J_0(\pi m_b(l)\sqrt{3}a) \\
\frac{\mathrm{d}y_2}{\mathrm{d}l} &= \left(2-\frac{1}{2}K_a-\frac{3}{2}K_b\right)y_2 - y_1 y_2 J_0\left(\pi\frac{\sqrt{3}}{2}m_b(l)a\right) \\
\frac{\mathrm{d}h}{\mathrm{d}l} &= \sqrt{\frac{3}{8a}}y_2^2 J_1(\pi\sqrt{6}m_b(l)a).
\end{aligned} \tag{C.3}$$

Appendix D. Determination of the exponents of the bosonized Hamiltonian

In this section, we discuss the determination of the exponents of the spin tube. We have shown previously that for weak coupling the model flows to a two-component Luttinger liquid fixed point. For strong coupling some alternative techniques are needed to determine the Luttinger liquid exponents from thermodynamic quantities. Note that in the isotropic SU(N) Hubbard model case [41] with a charge gap, one needs only the spin velocity since the spin exponents are constrained by SU(N) invariance. Here, SU(3) symmetry is broken and we expect two different velocities u_a, u_b and two exponents K_a, K_b, so we need four independent quantities.

Suppose that we have a two-component Luttinger liquid described by the Hamiltonian

$$H = \sum_{i=a,b} \int \frac{\mathrm{d}x}{2\pi} \left[u_i K_i (\pi \Pi_i)^2 + \frac{u_i}{K_i} (\partial_x \phi_i)^2 \right]. \tag{D.1}$$

In the case of the spin tube, we can rule out terms of the form $\Pi_a \Pi_b$ and $\partial_x \phi_a\, \partial_x \phi_b$ since we know that they are not present in the bare Hamiltonian and not generated by the RG. Moreover, we know that $\chi_{38} = 0$ which guarantees the absence of terms of the form $\partial_x \phi_a\, \partial_x \phi_b$ in the Hamiltonian.

The usual technique for determining the Luttinger liquid exponents is considering the energy change induced by taking $\pi \langle \Pi_{a,b} \rangle = \varphi_{a,b}/L$. In terms of the Luttinger liquid parameters, this energy change is given by

$$\delta E = \frac{u_a K_a}{2\pi L} (\varphi_a)^2 + \frac{u_b K_b}{2\pi L} (\varphi_b)^2. \tag{D.2}$$

This energy change is related to the change of ground-state energy caused by taking twisted boundary conditions. Let us discuss in more detail these twisted boundary conditions in the specific case of the spin tube. Using the bosonization formulae, one sees easily that

$$\begin{aligned} &\Lambda^1 + \mathrm{i}\Lambda^2 \propto \mathrm{e}^{\mathrm{i}\sqrt{2}\theta_a} \\ &\Lambda^4 + \mathrm{i}\Lambda^5 \propto \mathrm{e}^{\mathrm{i}(\theta_a/\sqrt{2} + \sqrt{2/3}\theta_b)} \\ &\Lambda^6 + \mathrm{i}\Lambda^7 \propto \mathrm{e}^{\mathrm{i}(-\theta_a/\sqrt{2} + \sqrt{2/3}\theta_b)}. \end{aligned} \tag{D.3}$$

Therefore, imposing $\langle \pi \Pi_a \rangle = \varphi_a/L$ and $\langle \pi \Pi_b \rangle = \varphi_b/L$ amounts to imposing the boundary conditions

$$\begin{aligned} &(\Lambda^1 + \mathrm{i}\Lambda^2)(L) = (\Lambda^1 + \mathrm{i}\Lambda^2)(0)\mathrm{e}^{\mathrm{i}\sqrt{2}\varphi_a} \\ &(\Lambda^4 + \mathrm{i}\Lambda^5)(L) = (\Lambda^4 + \mathrm{i}\Lambda^5)(0)\mathrm{e}^{\mathrm{i}(\varphi_a/\sqrt{2} + \sqrt{2/3}\varphi_b)} \\ &(\Lambda^6 + \mathrm{i}\Lambda^7)(L) = (\Lambda^6 + \mathrm{i}\Lambda^7)(0)\mathrm{e}^{\mathrm{i}(\sqrt{2/3}\varphi_b - \varphi_a/\sqrt{2})}. \end{aligned} \tag{D.4}$$

As an aside, one should remark that the transformation $\Pi(x) \to \Pi(x) - f(x)$ is realized by the operator

$$U = \exp\left(-\mathrm{i} \int \mathrm{d}x\ f(x) \phi(x) \right).$$

This operator can also be written as

$$U = \exp\left(\mathrm{i} \int \mathrm{d}x\ F(x)\, \partial_x \phi(x) \right)$$

where $f = \mathrm{d}F/\mathrm{d}x$. Twisted boundary conditions correspond to $f(x) = \alpha/L$. Therefore, an operator generating states satisfying boundary conditions (D.4) acting on states satisfying periodic boundary conditions can be built in the continuum. A lattice version is easily constructed, giving an operator of the form

$$U(\varphi) = \exp\left(-\mathrm{i} \sum_{n=1}^{L} \frac{(n-1)}{L} (\varphi_a \Lambda_n^3 + \varphi_b \Lambda_n^8) \right). \tag{D.5}$$

One can check that these lattice operators acting on states that satisfy periodic boundary conditions generate states that satisfy the boundary conditions (D.4) directly on the lattice. This guarantees the existence of states satisfying the twisted boundary conditions (D.4). The generalization of this construction to the SU(N) case is trivial. Instead of $\Lambda^{3,8}$ one has to consider the maximal Abelian subalgebra (MASA) and builds the operators corresponding to $U(\varphi)$. In the case of the spin tube, the energy change of the ground state obeys the condition

$$L\,\delta E(\varphi_a,\varphi_b) = L\,\delta E(\varphi_a,0) + L\,\delta E(0,\varphi_b) + \mathrm{o}(\varphi_a^2,\varphi_b^2). \tag{D.6}$$

In order to determine completely u_a, u_b, K_a, K_b, suppose that one places the anisotropic SU(3) chain in fields that couple to the Λ^3 and Λ^8 components of the spin:

$$H = \sum_i \sum_\alpha a_\alpha \lambda_i^\alpha \lambda_{i+1}^\alpha - h_3 \sum_i \lambda_i^3 - h_8 \sum_i \lambda_i^8. \tag{D.7}$$

Then, the Hamiltonian becomes in the continuum

$$H = H_0 + \sum_{\nu=a,b} h_\nu \frac{\partial_x \phi_\nu}{\pi} \tag{D.8}$$

where $h_a = h_3$, $h_b = h_8$. Then, one has

$$\frac{-\langle \partial_x \phi_\nu \rangle}{\pi} = \frac{K_\nu}{u_\nu} h_\nu. \tag{D.9}$$

Thus, one has

$$\begin{aligned} \langle \Lambda^3 \rangle &= K_a/u_a h_3 \\ \langle \Lambda^8 \rangle &= K_b/u_b h_8. \end{aligned} \tag{D.10}$$

This is sufficient for extracting the parameters of the two-component Luttinger liquid associated with the anisotropic SU(3) spin chain. However, we started from three coupled spin-1/2 chains with periodic boundary conditions. To extract the two-component Luttinger liquid exponents for this problem, we need to express the preceding formulae in terms of the original spins. Re-expressing the twisted boundary conditions in terms of the original spins is elementary if one remembers that when we choose $S^z = (5/6 - \lambda^8/\sqrt{3})$, we have

$$\begin{aligned} (\lambda^6 - \mathrm{i}\lambda^7)_i &= \frac{2}{\sqrt{3}} \sum_p j^{p-1} S_{i,p}^+ \\ (\lambda^4 - \mathrm{i}\lambda^5)_i &= \frac{2}{\sqrt{3}} \sum_p j^{2(p-1)} S_{i,p}^+ \\ (\lambda^1 - \mathrm{i}\lambda^2)_i &= -\frac{1}{2} \sum_p j^{2(p-1)} S_{i,p}^z. \end{aligned} \tag{D.11}$$

The following expression for λ^3 can also be obtained:

$$\lambda_3 = \frac{\mathrm{i}}{\sqrt{3}} \big((S_2^- S_1^+ - S_2^- S_1^+) + (S_3^- S_2^+ - S_3^+ S_2^-) + (S_1^- S_3^+ - S_1^+ S_3^-) \big) P_{S^z=1/2} \tag{D.12}$$

where $P_{S^z=1/2}$ is the projector on the subspace $S^z = 1/2$. Physically, λ_3 is proportional to the spin current in the transverse direction. It is therefore +1 for positive chirality and -1 for negative chirality. It can also be rewritten as

$$\lambda_3 = \frac{2}{\sqrt{3}} \left[(\vec{S}_2 \times \vec{S}_1)^z + (\vec{S}_3 \times \vec{S}_2)^z + (\vec{S}_1 \times \vec{S}_3)^z \right] P_{S^z=1/2}. \tag{D.13}$$

With these expressions, it is in principle possible to obtain numerically the Luttinger liquid exponents for a general three-leg spin ladder with periodic boundary conditions under a magnetic field between the $\langle S^z \rangle = 1/2$ and $\langle S^z \rangle = 3/2$ plateaus.

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